Efficient Thermodynamic Description of Multi-Component One-Dimensional Gases

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- 1 component and 2 component Bose gases with δ-function interaction
- thermodynamical Bethe ansatz
- Bose gases as continuum limits of lattice models:

  XXZ chain

  Uimin-Sutherland model
- path integral formalism for lattice model, quantum transfer matrix
- largest eigenvalue analysis + continuum limit $\rightarrow$ finite number of non-linear integral equations
- excited states, correlation lengths
- numerical results

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Integrable multi-component Bose gas

Hamiltonian for n-component gas

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\sum_{1\le i\le j\le N}\delta(x_i - x_j) - \sum_{i=1}^{n}\mu_i N_i\,.$$

$g = \hbar^2 c/m$: coupling constant, μ_i: chemical potentials

$n = 1$ scalar Bethe ansatz Lieb, Liniger 1963; McGuire 1964

$$e^{\mathrm{i}k_j L} = -\prod_{l=1}^{N}\frac{k_j - k_l + \mathrm{i}c}{k_j - k_l - \mathrm{i}c}$$

$n = 2$ nested Bethe ansatz Yang 1967; Sutherland 1968

$$e^{\mathrm{i}k_j L} = -\prod_{l=1}^{N}\frac{k_j - k_l + \mathrm{i}c}{k_j - k_l - \mathrm{i}c}\prod_{\alpha=1}^{M}\frac{k_j - \lambda_a - \frac{\mathrm{i}c}{2}}{k_j - \lambda_a + \frac{\mathrm{i}c}{2}},$$

$$\prod_{l=1}^{N}\frac{\lambda_\alpha - k_l - \frac{\mathrm{i}c}{2}}{\lambda_\alpha - k_l + \frac{\mathrm{i}c}{2}} = -\prod_{\beta=1}^{M}\frac{\lambda_\alpha - \lambda_\beta - \mathrm{i}c}{\lambda_\alpha - \lambda_\beta + \mathrm{i}c},$$

Thermodynamics: TBA

systems are thermodynamically unstable for attractive interaction c

For repulsive interactions ($c > 0$):

$n = 1$ **no bound states**

TBA for one function: Yang, Yang 1969

thermodynamical potential density

$$g = -T \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \ln\left[1 + e^{-\varepsilon(\lambda)/T}\right]$$

with dressed energy $\varepsilon(\lambda)$

$$\varepsilon(\lambda)/T = (\lambda^2 - \mu)/T - \kappa_2 * \ln\left[1 + e^{-\varepsilon(\lambda)/T}\right], \qquad \kappa_2(\lambda) = \frac{1}{\pi}\frac{c}{\lambda^2 + c^2}$$

N.b.: for nested case generalization

$$\kappa_j(\lambda) = \frac{1}{\pi}\frac{jc/2}{\lambda^2 + (jc/2)^2}$$

Density-density correlations: single component Bose gas

Seel, Bhattacharyya, Göhmann, AK 2007: multiple integrals in continuum limit of XXZ

(based on quantum transfer matrix, QTM)

Seel, Göhmann, AK 2008: Fredholm determinant for $c \to \infty$, explicit results

Kozlowski, Maillet, Slavnov 2010:

multiple integral formula for density-density correlations directly for continuum Bose gas

all c: asymptotic expansion of Fredholm determinants
$\simeq$ sum over all states of QTM

Kozlowski, Maillet, Slavnov 2011: CFT picture at low temperatures

Thermodynamics: TBA for two-component Bose gas

$n = 2$ ∞-many bound states Takahashi (1971)

thermodynamical potential

$$g = -T \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \ln\left[1 + e^{-\varepsilon(\lambda)/T}\right]$$

∞-many integral equations $\mu = (\mu_1 + \mu_2)/2,\ \Omega = (\mu_1 - \mu_2)/2$

$$\begin{aligned}
\varepsilon(\lambda)/T &= (\lambda^2 - \mu - \Omega)/T - \kappa_2 * \ln\left[1 + e^{-\varepsilon(\lambda)/T}\right] - \sum_{j=1}^{\infty} \kappa_j * \ln\left[1 + e^{-\varepsilon_j(\lambda)/T}\right] \\
\varepsilon_1(\lambda)/T &= f * \ln\left[1 + e^{-\varepsilon(\lambda)/T}\right] + f * \ln\left[1 + e^{\varepsilon_2(\lambda)/T}\right], \qquad f(\lambda) = \frac{1}{2c\cosh(\pi\lambda/c)} \\
\varepsilon_j(\lambda)/T &= f * \ln\left[1 + e^{\varepsilon_{j-1}(\lambda)/T}\right] + f * \ln\left[1 + e^{\varepsilon_{j+1}(\lambda)/T}\right], \qquad (j > 1),
\end{aligned}$$

with asymptotic condition

$$\lim_{j\to\infty} \frac{\varepsilon_j(\lambda)}{j} = 2\Omega$$

Gu, Li, Ying, Zhao (2002)

Thermodynamics: two-component Bose gas

$n = 2$ previous results

Eisenberg, Lieb (2002):

'ferromagnetism' for spin-independent interacting multi-component Bose gases

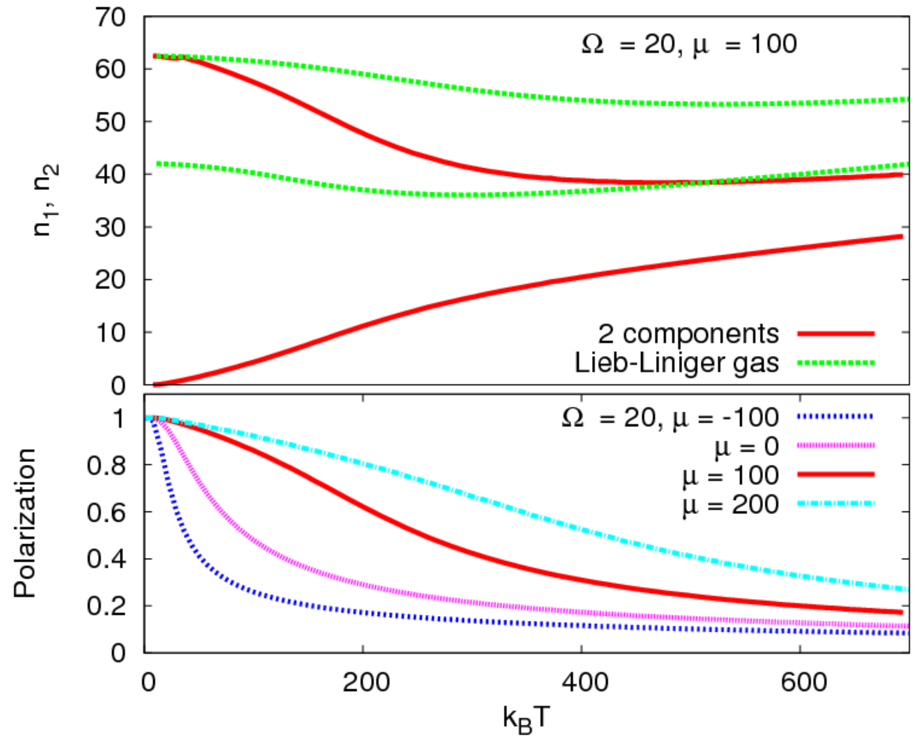
magnetic field $\Omega = (\mu_1 - \mu_2)/2$, polarization $P = (n_1 - n_2)/2$

Guan, Batchelor, Takahashi (2007): analytical low-temperature asymptotics

Caux, Klauser, van den Brink (2009, 2011):

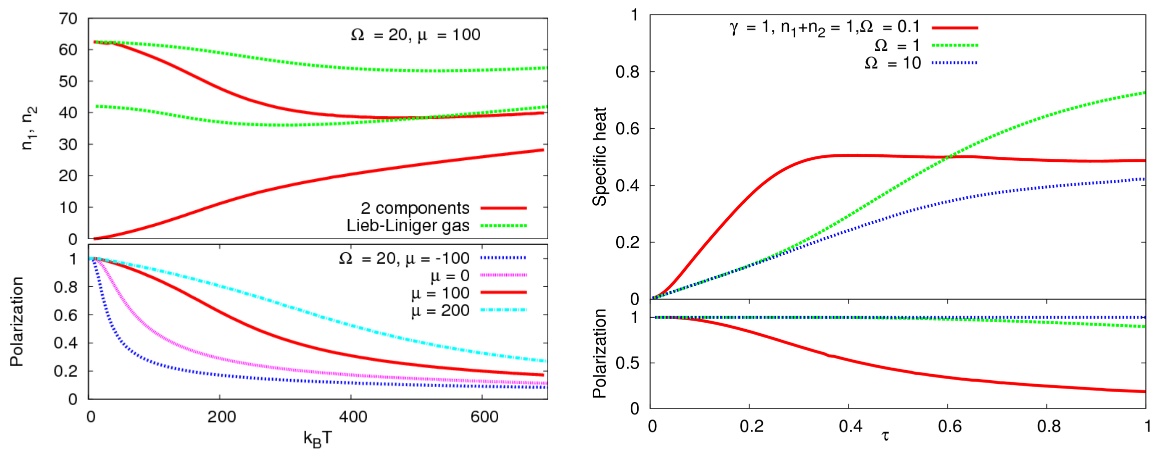
numerical solution of TBA for relatively large Ω

$(\hbar = 1,\ 2m = 1)$



TBA for two-component Bose gas

Caux, Klauser, van den Brink (2009, 2011):



TBA problematic at low T and small Ω:

similar to problems appearing in ferromagnetic spin-1/2 Heisenberg TBA

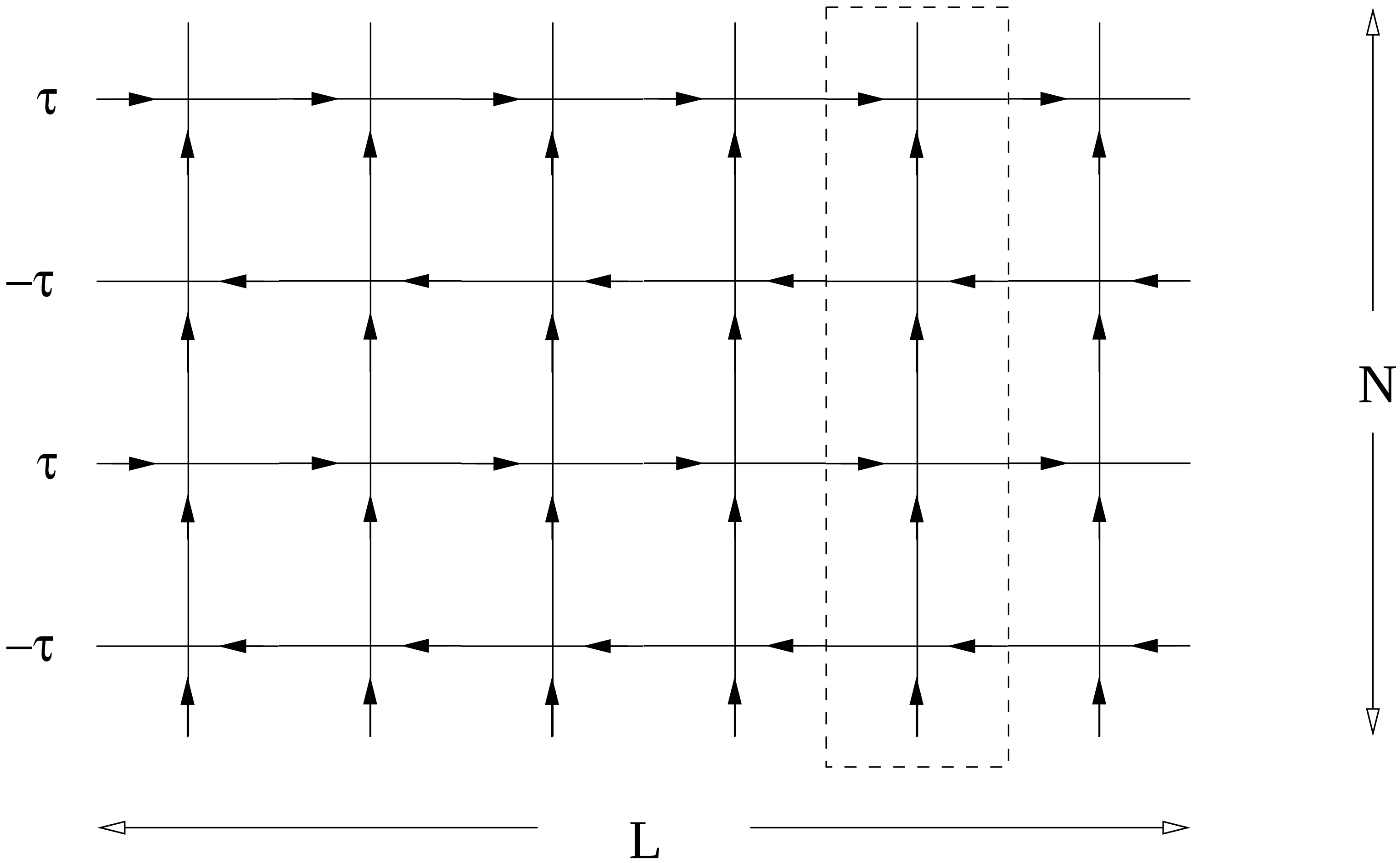
Here: search for alternative approach, finite number of equations

by analysis of largest eigenvalue of quantum transfer matrix

different set(s) of non-linear integral equations

Problem: quantum transfer matrix does not exist for continuum models

$\mathrm{Tr} e^{-\beta H} = \mathrm{Tr}\, T_{QTM}^{L} =$



$\tau = \beta / N$

free energy from largest eigenvalue of column-to-column transfer matrix (quantum transfer matrix, QTM)

Continuum limit of the spin-1/2 Heisenberg I

Continuum limit of the spin-1/2 XXZ chain close to the ferromagnetic XXX point: one- and two-particle data.
One-particle energy and momentum (with scales J and $1/\delta$ =1/lattice constant) and two-particle scattering:

$$E_m(v) = \frac{2J\mathrm{sh}^2\eta}{\mathrm{sh}(v+\eta/2)\mathrm{sh}(v-\eta/2)} + h \quad , \quad P_m(v) = -\frac{\mathrm{i}}{\delta}\ln\frac{\mathrm{sh}(v-\eta/2)}{\mathrm{sh}(v+\eta/2)}, \qquad S(v_j,v_l) = \frac{\mathrm{sh}(v_j-v_l-\eta)}{\mathrm{sh}(v_j-v_l+\eta)}$$

For small v and ε where $\eta = \mathrm{i}\gamma\,(\gamma = \pi - \varepsilon)$:

$$E_m(v) = 2J\delta^2\left[\left(\frac{\varepsilon v}{\delta}\right)^2 - \left(\frac{\varepsilon}{\delta}\right)^2\right] + h, \qquad P_m(v) = \frac{\varepsilon}{\delta}v, \qquad S(v_j,v_l) = \frac{\frac{\varepsilon}{\delta}v_j - \frac{\varepsilon}{\delta}v_l + \mathrm{i}\frac{\varepsilon^2}{\delta}}{\frac{\varepsilon}{\delta}v_j - \frac{\varepsilon}{\delta}v_l - \mathrm{i}\frac{\varepsilon^2}{\delta}}$$

Now we demand for the continuum limit ($\delta = l/L \to 0$) the finiteness of

$$\frac{\varepsilon}{\delta}v = k, \qquad 2J\delta^2 = 1\left(= \frac{1}{2m_B}\right), \qquad \frac{\varepsilon^2}{\delta} = c, \qquad \left(\frac{\varepsilon}{\delta}\right)^2 - h = \mu$$

Yields one and two-particle data of single component Bose gas.

Continuum limit of the spin-1/2 Heisenberg II

Continuum limit of the spin-1/2 XXZ chain close to the ferromagnetic XXX point

XXZ chain (5 parameters)	Bose gas (4 parameters)
interaction strength $J > 0$	particle mass $m_B = 1/(4J\delta^2)$ $(= 1/2)$
number of lattice sites L, lattice constant δ	physical length $\ell = L\delta$
magnetic field $h > 0$	chemical potential $\mu = 2J\varepsilon^2 - h$
anisotropy $\Delta = \varepsilon^2/2 - 1$	repulsion strength $c = \varepsilon^2/\delta$

(Seel, Bhattacharyya, Göhmann, AK (2007))

Thermodynamical potential per lattice site

$$\frac{F_{XXZ}}{L} = \frac{h}{2} - \int_C \frac{dw}{2\pi \mathrm{i}} \frac{\mathrm{sh}\eta}{\mathrm{sh}w\,\mathrm{sh}(w+\eta)} \ln(1+\mathfrak{a}(w))$$

with auxiliary function $\mathfrak{a}(v)$ from the non-linear integral equation

$$\ln \mathfrak{a}(v) = -\frac{h}{T} + \frac{2J\,\mathrm{sh}^2(\mathrm{i}\varepsilon)}{T\,\mathrm{sh}v\,\mathrm{sh}(v-\mathrm{i}\varepsilon)} + \int_C \frac{dw}{2\pi \mathrm{i}} \frac{\mathrm{sh}(2\mathrm{i}\varepsilon)}{\mathrm{sh}(v-w+\mathrm{i}\varepsilon)\mathrm{sh}(v-w-\mathrm{i}\varepsilon)} \ln(1+\mathfrak{a}(w))$$

continuum limit “2→1”: yields density of thermodynamical potential of single component Bose gas
Seel, Bhattacharyya, Göhmann, AK 2007: free energy and correlation functions!
Seel, Göhmann, AK 2008: Fredholm determinant

Continuum limit of the 3-state Uimin-Sutherland model

Quantum chain: Uimin-Sutherland model (1970, 1975)$\leftrightarrow$ classical model: Perk-Schultz model (1981)

$$R^{\alpha\alpha}_{\alpha\alpha}(v)=\frac{\sin(\gamma+\varepsilon_\alpha v)}{\sin\gamma}\qquad R^{\alpha\beta}_{\alpha\beta}(v)=\varepsilon_{\alpha\beta}\frac{\sin v}{\sin\gamma},\quad (\alpha\neq\beta)\qquad R^{\beta\alpha}_{\alpha\beta}(v)=e^{i\,\mathrm{sign}(\alpha-\beta)v},\quad (\alpha\neq\beta).$$

The largest eigenvalue of the QTM for the n-state model has the form $\Lambda_{QTM}(v)=\sum_{j=1}^{n}\lambda_j(v)$ with

$$\lambda_j(v)=\phi_-(v)\phi_+(v)\frac{q_{j-1}(v-i\varepsilon_j\gamma)}{q_{j-1}(v)}\frac{q_j(v+i\varepsilon_j\gamma)}{q_j(v)}e^{\beta\mu_j}$$

where $\phi_\pm(v)=\left(\frac{\sinh(v\pm iu)}{\sin\gamma}\right)^{N/2}$, $u=\frac{\beta}{N}$ and $q_j(v)=\begin{cases}\phi_-(v) & j=0\\ \prod_{k=1}^{N/2}\sinh(v-v_k^{(j)}) & j=1,2,\cdots,n-1\\ \phi_+(v) & j=n\end{cases}$

Transformation to non-linear integral equations:

(i) ∞-many of TBA type $\rightarrow$ still ∞-many functions in continuum limit,
(ii) finite number based on excitations on physical vacuum $\rightarrow$ even fewer functions in continuum limit,
(iii) Takahashi's 1999/2000 approach

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(i) Jüttner, AK, Suzuki 1998, (ii) Fujii, AK 1999

Derivation of integral equations

Two sets of Bethe ansatz equations for 3-state model (with $\gamma = \pi - \varepsilon$) written as two contour integrals

$$\log a_1(v) = \beta(\mu_1 - \mu_3) + \beta \frac{\mathrm{sh}^2(\mathrm{i}\varepsilon)}{\mathrm{sh} v\, \mathrm{sh}(v-\mathrm{i}\varepsilon)} + \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{sh}(2\mathrm{i}\varepsilon)}{\mathrm{sh}(v-w-\mathrm{i}\varepsilon)\mathrm{sh}(v-w+\mathrm{i}\varepsilon)} \log(1+a_1(w))dw$$
$$- \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{sh}(\mathrm{i}\varepsilon)}{\mathrm{sh}(v-w-\mathrm{i}\varepsilon)\mathrm{sh}(v-w)} \log(1+a_2(w))dw$$

$$\log a_2(v) = \beta(\mu_2 - \mu_3) + \beta \frac{\mathrm{sh}^2(\mathrm{i}\varepsilon)}{\mathrm{sh} v\, \mathrm{sh}(v+\mathrm{i}\varepsilon)} + \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{sh}(\mathrm{i}\varepsilon)}{\mathrm{sh}(v-w+\mathrm{i}\varepsilon)\mathrm{sh}(v-w)} \log(1+a_1(w))dw$$
$$- \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathrm{sh}(2\mathrm{i}\varepsilon)}{\mathrm{sh}(v-w-\mathrm{i}\varepsilon)\mathrm{sh}(v-w+\mathrm{i}\varepsilon)} \log(1+a_2(w))dw$$

where $a_1(v)$ and $a_2(v)$ are 'lhs/rhs' of the Bethe equations.

C is closed contour around real axis (in total 4 straight integration paths)

$\rightarrow$ in continuum limit only upper or lower part contributes $\rightarrow$ 2 equations

Integral equations for 2-component Bose gas

Thermodynamical potential density

$$g = -\frac{T}{2\pi}\int_{-\infty}^{\infty} d\lambda \ln[(1+a_1(\lambda))(1+a_2(\lambda))]$$

where a_1 and a_2 satisfy

$$\ln a_1 = -\beta(\lambda^2-\mu-\Omega)+\kappa_2 * \ln(1+a_1)+\kappa_1^+ * \ln(1+a_2),$$
$$\ln a_2 = -\beta(\lambda^2-\mu+\Omega)+\kappa_1^- * \ln(1+a_1)+\kappa_2 * \ln(1+a_2),$$

where

$$\kappa_2(\lambda) = \frac{1}{\pi}\frac{c}{\lambda^2+c^2}. \qquad \kappa_1(\lambda) = \frac{1}{\pi}\frac{c/2}{\lambda^2+(c/2)^2}, \qquad \kappa_1^{\pm}(\lambda) = \kappa_1(\lambda \pm \mathrm{i}c/2)$$

(AK, Patu 2011)

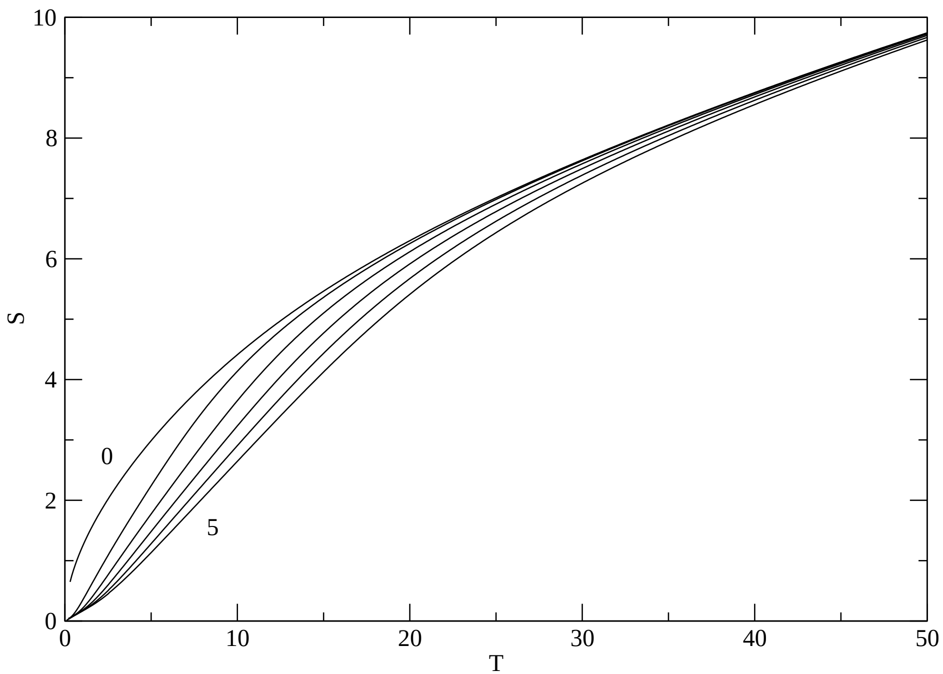
Results: entropy, specific heat

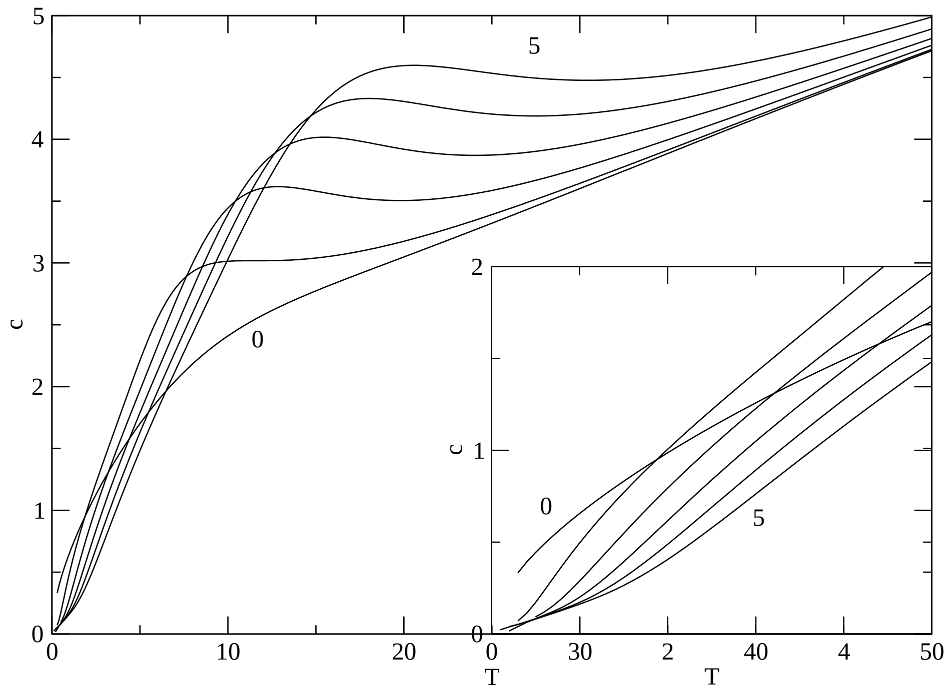
interaction $c = 1$

chemical potentials $\mu_1 = 15 + \Omega$, $\mu_2 = 15 - \Omega$

$\Omega = 0, 1, 2, 3, 4, 5$

entropy and specific heat





Note: square root dependence on T for $\Omega = 0$, linear behaviour for $\Omega \neq 0$

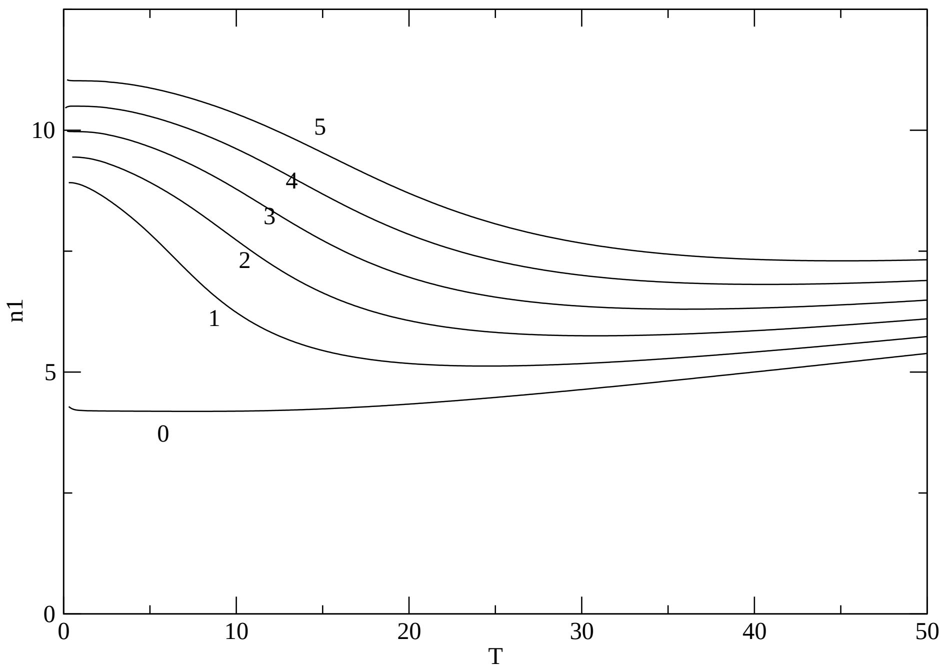
Results: particle densities

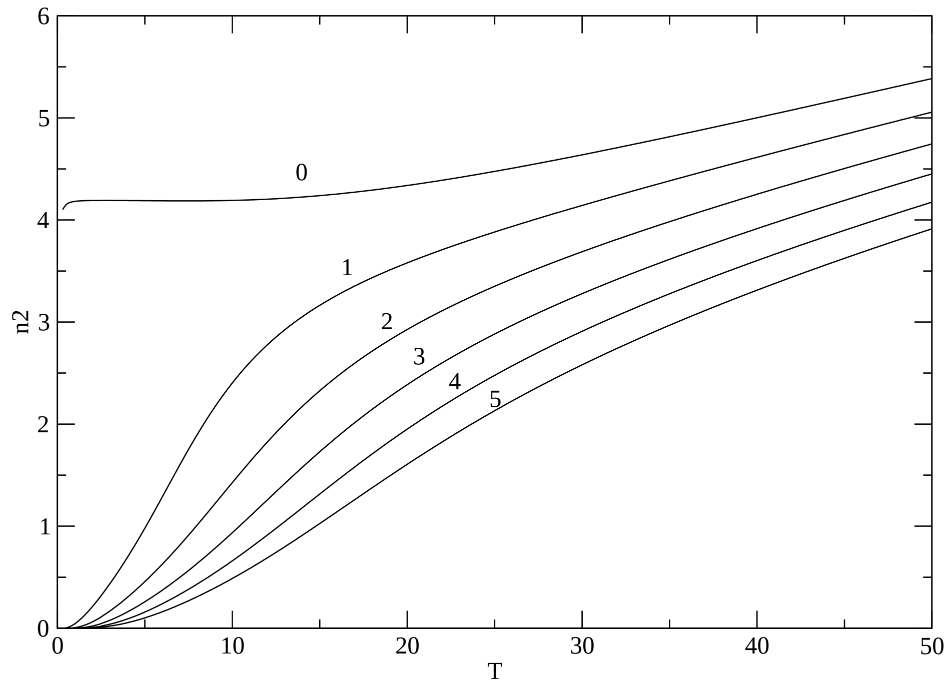
interaction $c = 1$

chemical potentials $\mu_1 = 15 + \Omega$, $\mu_2 = 15 - \Omega$

$\Omega = 0, 1, 2, 3, 4, 5$

particle densities n_1, n_2





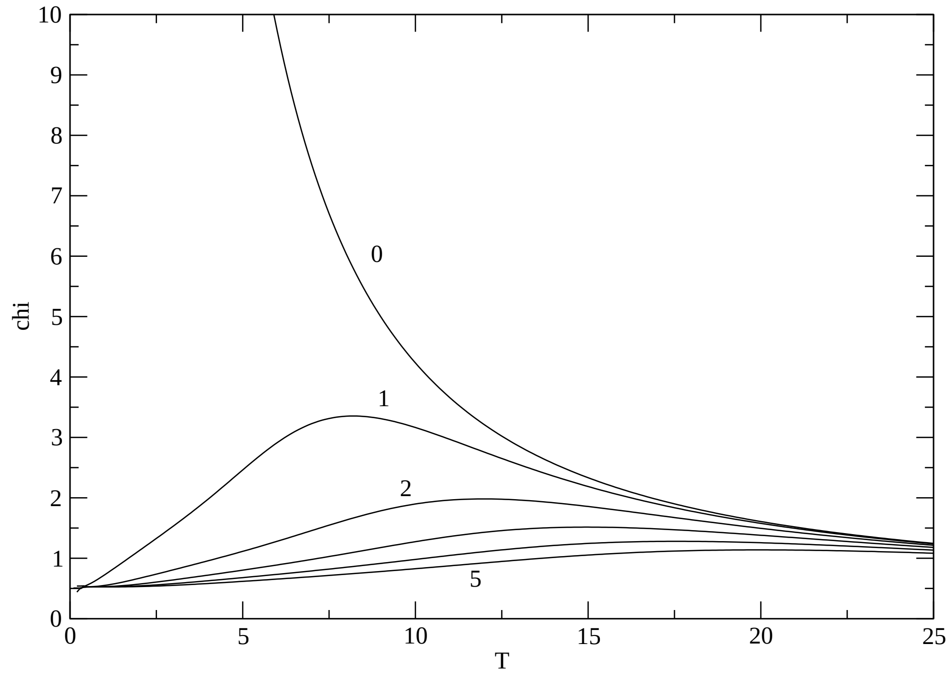
Note: continuous dependence on Ω for $T > 0$, jump at $\Omega = 0$ for $T = 0$

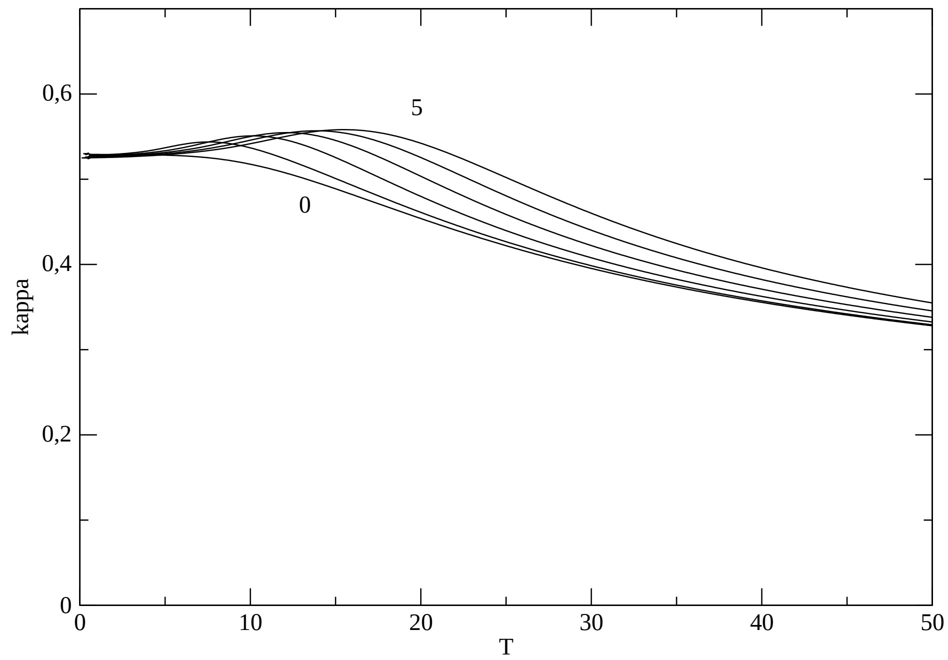
Results: susceptibilities

interaction $c = 1$

chemical potentials $\mu_1 = 15 + \Omega$, $\mu_2 = 15 - \Omega$

$\Omega = 0, 1, 2, 3, 4, 5$ magnetic and particle susceptibilities χ, κ





Note: for $\Omega = 0$: divergence of χ for $\Omega \neq 0$: $\chi(0) = \kappa(0)$

Integral equations for excited states

Correlation length ξ of the field-field correlation function $\langle \Psi_1^\dagger(x)\Psi_1(0)\rangle \sim e^{-x/\xi}$

$$\frac{1}{\xi} = \ln\left(\frac{\Lambda_0}{\Lambda_1}\right)$$

where $\Lambda_{0,1}$ are the largest and next-largest eigenvalue of the “continuum" QTM.

We find $\ln\Lambda_1 = ik_0 + \frac{1}{2\pi}\int \ln[A_1(k)A_2(k)]\,dk$, where

$$\begin{aligned}
\ln a_1 &= -\beta(\lambda^2-\mu-\Omega)+\ln\left(\frac{k-k_0+ic}{k-k_0-ic}\right) \quad +\kappa_2 * \ln(1+a_1)+\kappa_1^+ * \ln(1+a_2),\\
\ln a_2 &= -\beta(\lambda^2-\mu+\Omega) \quad +\kappa_1^- * \ln(1+a_1)+\kappa_2 * \ln(1+a_2),
\end{aligned}$$

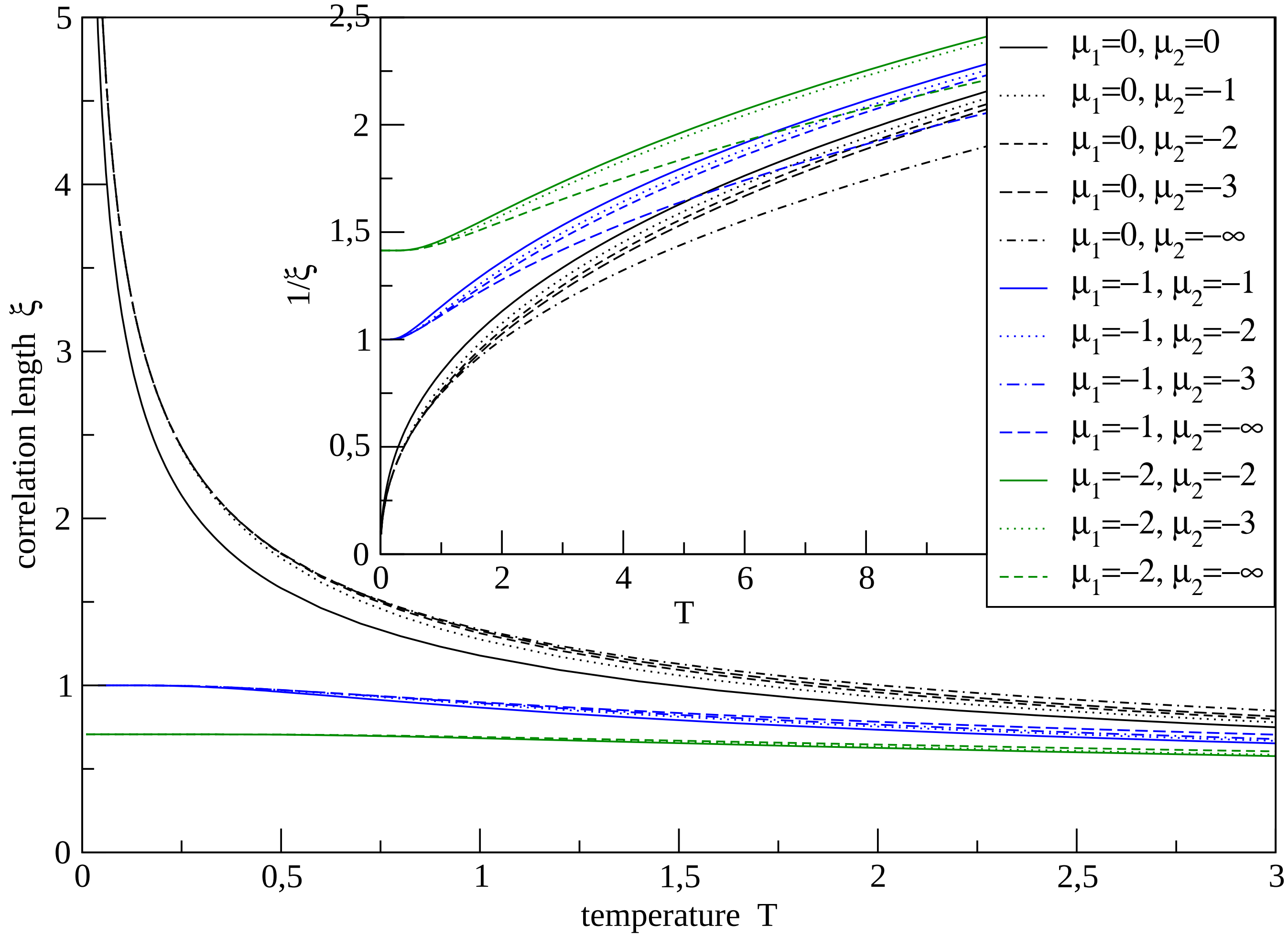
The rapidity k_0 is subject to the condition $1+a_1(k_0)=0$

appears to be purely imaginary in the dilute gas phase (μ_1, $\mu_2 \leq 0$).

(AK, Patu 2011)

Field-field correlators: correlation lengths

Correlation length ξ of the Green's function for particles of type 1 (interaction $c = 1$)



$T \to 0$ behaviour of ξ: finite for $\mu_1 < 0$ and $T^{-1/2}$ divergent for $\mu_1 = 0$.

Summary and outlook

- (i) study of 2-component Bose gas with δ-function interaction

  (ii) new lattice embedding of the Bose gas

  (iii) derivation of alternative thermodynamical equations: closed set of just 2 equations

  (iv) correlation lengths for field-field correlators

  (v) numerical study

- Outlook

  (i) more than 2 components?

  (ii) other correlation lengths?

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