Efficient Thermodynamic Description of Multi-Component One-Dimensional Gases

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• 1 component and 2 component Bose gases with $\delta$-function interaction

• thermodynamical Bethe ansatz

• Bose gases as continuum limits of lattice models:
  - $XXZ$ chain
  - Uimin-Sutherland model

• path integral formalism for lattice model, quantum transfer matrix

• largest eigenvalue analysis + continuum limit $\rightarrow$ finite number of non-linear integral equations

• excited states, correlation lengths

• numerical results

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Integrable multi-component Bose gas

Hamiltonian for $n$-component gas

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \sum_{1 \leq i \leq j \leq N} \delta(x_i - x_j) - \sum_{i=1}^{n} \mu_i N_i.$$ 

$g = \hbar^2 c/m$: coupling constant, $\mu_i$: chemical potentials

$n = 1$ scalar Bethe ansatz

$$e^{ik_jL} = -\prod_{l=1}^{N} \frac{k_j - k_l + ic}{k_j - k_l - ic}.$$ 

$n = 2$ nested Bethe ansatz

$$e^{ik_jL} = -\prod_{l=1}^{N} \frac{k_j - k_l + ic}{k_j - k_l - ic} \prod_{a=1}^{M} \frac{k_j - \lambda_a - \frac{ic}{2}}{k_j - \lambda_a + \frac{ic}{2}},$$ 

$$\prod_{l=1}^{N} \frac{\lambda_\alpha - k_l - \frac{ic}{2}}{\lambda_\alpha - k_l + \frac{ic}{2}} = -\prod_{\beta=1}^{M} \frac{\lambda_\alpha - \lambda_\beta - ic}{\lambda_\alpha - \lambda_\beta + ic},$$
Thermodynamics: TBA

systems are thermodynamically unstable for attractive interaction $c$

For repulsive interactions ($c > 0$):

$n = 1$ no bound states

TBA for one function:

thermodynamical potential density

$$g = -T \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \ln \left[ 1 + e^{-\epsilon(\lambda)/T} \right]$$

with dressed energy $\epsilon(\lambda)$

$$\epsilon(\lambda)/T = (\lambda^2 - \mu)/T - \kappa_2 * \ln \left[ 1 + e^{-\epsilon(\lambda)/T} \right], \quad \kappa_2(\lambda) = \frac{1}{\pi} \frac{c}{\lambda^2 + c^2}$$

N.b.: for nested case generalization

$$\kappa_j(\lambda) = \frac{1}{\pi} \frac{jc/2}{\lambda^2 + (jc/2)^2}$$

Yang, Yang 1969
Density-density correlations: single component Bose gas

Seel, Bhattacharyya, Göhmann, AK 2007: multiple integrals in continuum limit of $XXZ$
(based on quantum transfer matrix, QTM)

Seel, Göhmann, AK 2008: Fredholm determinant for $c \to \infty$, explicit results

Kozlowski, Maillet, Slavnov 2010:
multiple integral formula for density-density correlations directly for continuum Bose gas
all $c$: asymptotic expansion of Fredholm determinants
$\simeq$ sum over all states of QTM

Kozlowski, Maillet, Slavnov 2011: CFT picture at low temperatures
\[ n = 2 \text{-} \text{many bound states} \]

Thermodynamical potential

\[ g = -T \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \ln \left[ 1 + e^{-\varepsilon(\lambda)/T} \right] \]

\[ \varepsilon(\lambda)/T = (\lambda^2 - \mu - \Omega)/T - \kappa_2 \ln \left[ 1 + e^{-\varepsilon(\lambda)/T} \right] - \sum_{j=1}^{\infty} \kappa_j \ln \left[ 1 + e^{-\varepsilon_j(\lambda)/T} \right] \]

\[ \varepsilon_1(\lambda)/T = f \ln \left[ 1 + e^{-\varepsilon(\lambda)/T} \right] + f \ln \left[ 1 + e^{\varepsilon_2(\lambda)/T} \right] \]

\[ \varepsilon_j(\lambda)/T = f \ln \left[ 1 + e^{\varepsilon_{j-1}(\lambda)/T} \right] + f \ln \left[ 1 + e^{\varepsilon_{j+1}(\lambda)/T} \right], \quad (j > 1), \]

with asymptotic condition

\[ \lim_{j \to \infty} \frac{\varepsilon_j(\lambda)}{j} = 2\Omega \]
**Thermodynamics: two-component Bose gas**

**n = 2 previous results**

Eisenberg, Lieb (2002):
‘ferromagnetism’ for spin-independent interacting multi-component Bose gases
magnetic field $\Omega = (\mu_1 - \mu_2)/2$, polarization $P = (n_1 - n_2)/2$

Guan, Batchelor, Takahashi (2007): analytical low-temperature asymptotics

Caux, Klauser, van den Brink (2009, 2011):
numerical solution of TBA for relatively large $\Omega$
($\hbar = 1$, $2m = 1$)
Caux, Klauser, van den Brink (2009, 2011):

**TBA problematic at low $T$ and small $\Omega$:**

similar to problems appearing in ferromagnetic spin-1/2 Heisenberg TBA

**Here:** search for alternative approach, finite number of equations

by analysis of largest eigenvalue of quantum transfer matrix

different set(s) of non-linear integral equations

**Problem:** quantum transfer matrix does not exist for continuum models
$\text{Tr}e^{-\beta H} = \text{Tr} T_{QTM}^L = $

free energy from largest eigenvalue of column-to-column transfer matrix (quantum transfer matrix, QTM)

$\tau = \beta / N$
Continuum limit of the spin-1/2 Heisenberg I

Continuum limit of the spin-1/2 XXZ chain close to the ferromagnetic XXX point: one- and two-particle data. One-particle energy and momentum (with scales $J$ and $1/\delta = 1/l$ lattice constant) and two-particle scattering:

$$E_m(v) = \frac{2J\sh^2 \eta}{\sh(v + \eta/2)\sh(v - \eta/2)} + h, \quad P_m(v) = -\frac{i}{\delta} \ln \frac{\sh(v - \eta/2)}{\sh(v + \eta/2)}, \quad S(v_j, v_l) = \frac{\sh(v_j - v_l - \eta)}{\sh(v_j - v_l + \eta)}$$

For small $v$ and $\varepsilon$ where $\eta = i\gamma$ ($\gamma = \pi - \varepsilon$):

$$E_m(v) = 2J\delta^2 \left[ \left( \frac{\varepsilon}{\delta} \right)^2 - \left( \frac{\varepsilon}{\delta} \right)^2 \right] + h, \quad P_m(v) = \frac{\varepsilon}{\delta} v, \quad S(v_j, v_l) = \frac{\varepsilon}{\delta} v_j - \frac{\varepsilon}{\delta} v_l + i\frac{\varepsilon^2}{\delta}$$

Now we demand for the continuum limit ($\delta = l/L \to 0$) the finiteness of

$$\frac{\varepsilon}{\delta} v = k, \quad 2J\delta^2 = 1 \left( = \frac{1}{2m_B} \right), \quad \frac{\varepsilon^2}{\delta} = c, \quad \left( \frac{\varepsilon}{\delta} \right)^2 - h = \mu$$

Yields one and two-particle data of single component Bose gas.
Continuum limit of the spin-1/2 Heisenberg II

Continuum limit of the spin-1/2 XXZ chain close to the ferromagnetic XXX point

<table>
<thead>
<tr>
<th>XXZ chain (5 parameters)</th>
<th>Bose gas (4 parameters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>interaction strength $J &gt; 0$</td>
<td>particle mass $m_B = 1/(4J\delta^2)$ ($= 1/2$)</td>
</tr>
<tr>
<td>number of lattice sites $L$, lattice constant $\delta$</td>
<td>physical length $\ell = L\delta$</td>
</tr>
<tr>
<td>magnetic field $h &gt; 0$</td>
<td>chemical potential $\mu = 2J\varepsilon^2 - h$</td>
</tr>
<tr>
<td>anisotropy $\Delta = \varepsilon^2/2 - 1$</td>
<td>repulsion strength $c = \varepsilon^2/\delta$</td>
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(Seel, Bhattacharyya, Göhmann, AK (2007))

Thermodynamical potential per lattice site

$$\frac{F_{XXZ}}{L} = \frac{h}{2} - \int_{C} \frac{dw}{2\pi i} \frac{\text{sh}\eta}{\text{sh}\text{sh}(w + \eta)} \ln(1 + a(w))$$

with auxiliary function $a(v)$ from the non-linear integral equation

$$\ln a(v) = -\frac{h}{T} + \frac{2J\text{sh}^2(i\varepsilon)}{T\text{shvsh}(v - i\varepsilon)} + \int_{C} \frac{dw}{2\pi i} \frac{\text{sh}(2i\varepsilon)}{\text{sh}(v - w + i\varepsilon)\text{sh}(v - w - i\varepsilon)} \ln(1 + a(w))$$

continuum limit “2→1”: yields density of thermodynamical potential of single component Bose gas

Seel, Bhattacharyya, Göhmann, AK 2007: free energy and correlation functions!
Seel, Göhmann, AK 2008: Fredholm determinant
Continuum limit of the 3-state Uimin-Sutherland model


\[
R_{\alpha\alpha}^{\alpha\alpha}(v) = \frac{\sin(\gamma + \varepsilon_\alpha v)}{\sin \gamma}, \quad R_{\alpha\beta}^{\alpha\beta}(v) = \varepsilon_{\alpha\beta} \frac{\sin v}{\sin \gamma}, \quad (\alpha \neq \beta) \quad R_{\alpha\beta}^{\beta\alpha}(v) = e^{i \text{sign}(\alpha - \beta) v}, \quad (\alpha \neq \beta).
\]

The largest eigenvalue of the QTM for the \(n\)-state model has the form \(\Lambda_{\text{QTM}}(v) = \sum_{j=1}^{n} \lambda_j(v)\) with

\[
\lambda_j(v) = \phi_-(v) \phi_+(v) \frac{q_{j-1}(v - i \varepsilon_j \gamma)}{q_{j-1}(v)} \frac{q_j(v + i \varepsilon_j \gamma)}{q_j(v)} e^{\beta \mu_j}
\]

where \(\phi_{\pm}(v) = \left(\frac{\sinh(v \pm i u)}{\sin \gamma}\right)^{N/2}\), \(u = \frac{\beta}{N}\) and \(q_j(v) = \begin{cases} 
\phi_-(v) & j = 0 \\
\prod_{k=1}^{N/2} \sinh(v - v_k^{(j)}) & j = 1, 2, \cdots, n - 1 \\
\phi_+(v) & j = n
\end{cases}\)

Transformation to non-linear integral equations:

(i) \(\infty\)-many of TBA type \(\rightarrow\) still \(\infty\)-many functions in continuum limit,
(ii) finite number based on excitations on physical vacuum \(\rightarrow\) even fewer functions in continuum limit,
(iii) Takahashi’s 1999/2000 approach

\(\cdot\)
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Derivation of integral equations

Two sets of Bethe ansatz equations for 3-state model (with $\gamma = \pi - \varepsilon$) written as two contour integrals

$$\log a_1(v) = \beta(\mu_1 - \mu_3) + \beta \frac{\text{sh}^2(i\varepsilon)}{\text{sh}v \text{sh}(v-i\varepsilon)} + \frac{1}{2\pi i} \oint_C \frac{\text{sh}(2i\varepsilon)}{\text{sh}(v-w-i\varepsilon) \text{sh}(v-w+i\varepsilon)} \log(1 + a_1(w)) dw$$

$$- \frac{1}{2\pi i} \oint_C \frac{\text{sh}(i\varepsilon)}{\text{sh}(v-w-i\varepsilon) \text{sh}(v-w)} \log(1 + a_2(w)) dw$$

$$\log a_2(v) = \beta(\mu_2 - \mu_3) + \beta \frac{\text{sh}^2(i\varepsilon)}{\text{sh}v \text{sh}(v+i\varepsilon)} + \frac{1}{2\pi i} \oint_C \frac{\text{sh}(i\varepsilon)}{\text{sh}(v-w+i\varepsilon) \text{sh}(v-w)} \log(1 + a_1(w)) dw$$

$$- \frac{1}{2\pi i} \oint_C \frac{\text{sh}(2i\varepsilon)}{\text{sh}(v-w-i\varepsilon) \text{sh}(v-w+i\varepsilon)} \log(1 + a_2(w)) dw$$

where $a_1(v)$ and $a_2(v)$ are ‘lhs/rhs’ of the Bethe equations.

$C$ is closed contour around real axis (in total 4 straight integration paths)

$\rightarrow$ in continuum limit only upper or lower part contributes $\rightarrow$ 2 equations
Integral equations for 2-component Bose gas

Thermodynamical potential density

\[ g = -\frac{T}{2\pi} \int_{-\infty}^{\infty} d\lambda \ln[(1 + a_1(\lambda))(1 + a_2(\lambda))] \]

where \( a_1 \) and \( a_2 \) satisfy

\[
\ln a_1 = -\beta (\lambda^2 - \mu - \Omega) + \kappa_2 * \ln(1 + a_1) + \kappa_1^+ * \ln(1 + a_2),
\]

\[
\ln a_2 = -\beta (\lambda^2 - \mu + \Omega) + \kappa_1^- * \ln(1 + a_1) + \kappa_2 * \ln(1 + a_2),
\]

where

\[
\kappa_2(\lambda) = \frac{1}{\pi} \frac{c}{\lambda^2 + c^2}, \quad \kappa_1(\lambda) = \frac{1}{\pi} \frac{c/2}{\lambda^2 + (c/2)^2}, \quad \kappa_1^\pm(\lambda) = \kappa_1(\lambda \pm ic/2)
\]

(AK, Patu 2011)
interaction $c = 1$
chemical potentials $\mu_1 = 15 + \Omega$, $\mu_2 = 15 - \Omega$
$\Omega = 0, 1, 2, 3, 4, 5$

entropy and specific heat

Note: square root dependence on $T$ for $\Omega = 0$, linear behaviour for $\Omega \neq 0$
Results: particle densities

interaction $c = 1$

chemical potentials $\mu_1 = 15 + \Omega$, $\mu_2 = 15 - \Omega$

$\Omega = 0, 1, 2, 3, 4, 5$

particle densities $n_1, n_2$

Note: continuous dependence on $\Omega$ for $T > 0$, jump at $\Omega = 0$ for $T = 0$
interaction $c = 1$

chemical potentials $\mu_1 = 15 + \Omega, \mu_2 = 15 - \Omega$

$\Omega = 0, 1, 2, 3, 4, 5$ magnetic and particle susceptibilities $\chi, \kappa$

Note: for $\Omega = 0$: divergence of $\chi$ for $\Omega \neq 0$: $\chi(0) = \kappa(0)$
Integral equations for excited states

Correlation length $\xi$ of the field-field correlation function $\langle \Psi_1^\dagger(x)\Psi_1(0) \rangle \sim e^{-x/\xi}$

$$\frac{1}{\xi} = \ln \left( \frac{\Lambda_0}{\Lambda_1} \right)$$

where $\Lambda_{0,1}$ are the largest and next-largest eigenvalue of the “continuum” QTM.

We find $\ln \Lambda_1 = ik_0 + \frac{1}{2\pi} \int \ln[A_1(k)A_2(k)] dk$, where

$$\ln a_1 = -\beta(\lambda^2 - \mu - \Omega) + \ln \left( \frac{k - k_0 + ic}{k - k_0 - ic} \right) + \kappa_2 \ast \ln(1 + a_1) + \kappa_1^+ \ast \ln(1 + a_2),$$

$$\ln a_2 = -\beta(\lambda^2 - \mu + \Omega) + \kappa_1^- \ast \ln(1 + a_1) + \kappa_2 \ast \ln(1 + a_2),$$

The rapidity $k_0$ is subject to the condition $1 + a_1(k_0) = 0$

appears to be purely imaginary in the dilute gas phase ($\mu_1, \mu_2 \leq 0$).

(AK, Patu 2011)
Correlation length $\xi$ of the Green's function for particles of type 1 (interaction $c = 1$)

$T \to 0$ behaviour of $\xi$: finite for $\mu_1 < 0$ and $T^{-1/2}$ divergent for $\mu_1 = 0$. 
Summary and outlook

• (i) study of 2-component Bose gas with δ-function interaction
  (ii) new lattice embedding of the Bose gas
  (iii) derivation of alternative thermodynamical equations: closed set of just 2 equations
  (iv) correlation lengths for field-field correlators
  (v) numerical study

• Outlook
  (i) more than 2 components?
  (ii) other correlation lengths?