Current fluctuations and relaxation rates of the asymmetric exclusion process

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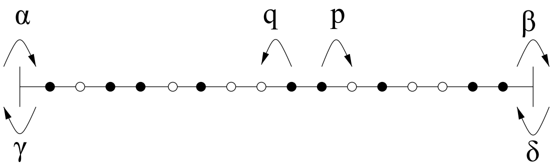
arXiv:1107.2744 (de Gier, Finn, Sorrell)

The Asymmetric Exclusion Process (ASEP)

Non-equilibrium system describing the diffusion of hard-core particles along a one-dimensional chain



We will make use of the parameterisation

$$a = \kappa^{+}_{\alpha,\gamma}, \quad b = \kappa^{+}_{\beta,\delta}, \quad c = \kappa^{-}_{\alpha,\gamma}, \quad d = \kappa^{-}_{\beta,\delta}$$

where $\kappa^{\pm}_{\alpha,\gamma}$ are the solutions of

$$\alpha\kappa^2_{\alpha,\gamma} + (p - q - \alpha - \gamma)\kappa_{\alpha,\gamma} + \gamma = 0$$

Transition Matrix

- Hopping rules encoded by the transition matrix

$$M = m_1 + \sum_{i=1}^{L-1} m_{i,i+1} + m_L$$

- e.g. left boundary

$$m_1 = \begin{array}{c} \\ |0\rangle \\ |1\rangle \end{array} \begin{array}{c} \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix} \end{array}$$

- Related to Hamiltonian of the XXZ spin chain with non-diagonal open boundaries by a similarity transformation

$$M \to H_{XXZ} = \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z + \text{b.t.}$$

Master Equation

Time evolution governed by a master equation:

- Probability distribution of configurations

$$|P(t)\rangle = \sum_{\boldsymbol{\tau}} P(\boldsymbol{\tau}|t)|\boldsymbol{\tau}\rangle$$

- Time evolution

$$\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = M|P(t)\rangle$$

- Formal solution

$$|P(t)\rangle = \exp(Mt)|P(0)\rangle$$

- Late time behaviour given by eigenvalues with largest real parts

$$|P(t)\rangle \simeq a_0|\psi_0\rangle + a_1 e^{\mathcal{E}_1 t}|\psi_1\rangle$$

where $M|\psi_0\rangle = 0 \quad M|\psi_1\rangle = \mathcal{E}_1|\psi_1\rangle$

Current Fluctuations

- Interested in the probabillity distribution of $Q_1(t)$, the total time-integrated current
- Calculate the quantity

$$E(\lambda) = \lim_{L\to\infty} \lim_{t\to\infty} \frac{1}{t} \log\langle e^{\lambda Q_1(t)}\rangle$$

- The generating function $\langle e^{\lambda Q_t(t)}\rangle$ encodes moments of the distribution

$\lim_{t\to\infty} \frac{\langle Q_1\rangle}{t}$	Bulk current
$\lim_{t\to\infty} \frac{\langle Q_1^2\rangle - \langle Q_1\rangle^2}{t}$	Fluctuations

Generalised transition matrix

- Introduce a fugacity e^{λ} at the first site

$$m_1 = \begin{pmatrix} -\alpha & \gamma e^{-\lambda} \\ \alpha e^{\lambda} & -\gamma \end{pmatrix}$$

- $E(\lambda)$ is the largest eigenvalue of the generalised "transition matrix" $M(\lambda)$
- Eigenvalues of $M(\lambda)$ from Bethe ansatz method when parameters satisfy

$$\left(q^{L/2+k} - e^{\lambda}\right)\left(e^{\lambda} - q^{L/2-k-1}abcd\right) = 0,$$

where $k \in \mathbb{Z}$, $|k| \leq L/2$ (and have set $p = 1$)
- Constrains λ to discrete sequence of values

The Bethe ansatz

- Two sets of Bethe equations for the open XXZ Hamiltonian
- To calculate $E(\lambda)$, require the lowest eigenvalue, given by

$$E(\lambda) = \sum_{j=1}^{L/2+k} \frac{(1-q)^2 z_j}{(1-z_j)(1-qz_j)} = \sum_{j=1}^{N} \varepsilon(z_j)$$

- $N = L/2 + k$ Bethe roots, z_j, solutions of Bethe equations

$$Y_L(z_j) = \frac{2\pi}{L} I_j, \quad j = 1, \ldots, \frac{L}{2} + k$$

- I_j are integers, Y_L is the counting function

$$iY_L(z) = g(z) + \frac{1}{L} g_b(z) + \frac{1}{L} \sum_{j=1}^{N} K(z_j, z)$$

- Eigenvalue of first excited state given by **second set**

Sequence (S1)

- Define the sequence (S1)

$$\lambda_n^{(1)} = n \ln(q), \quad n = 0, 1, 2 \dots$$

  satisfying first constraint with $k = -L/2 + n$

- For $\lambda = \lambda_n^{(1)}$, $N = n$ Bethe roots

$$z_j = -\frac{q^{j-1}}{a} + \mathcal{O}(e^{-\mu_j L}), \quad j = 1, \dots, n$$

- Ground state energy

$$E = \sum_{j=1}^{n} \varepsilon\left(-\frac{q^{j-1}}{a}\right) = (1-q)\left(\frac{a}{a+1} - \frac{a}{a+q^n}\right)$$

- Restoring p, $\lambda_n^{(1)}$ ($q^n = e^{\lambda_n^{(1)}}$)

$$E(\lambda_n^{(1)}) = (p-q)\frac{a(e^{\lambda_n^{(1)}} - 1)}{(1+a)(e^{\lambda_n^{(1)}} + a)}$$

Structure of the (S1) solution

- Counting function

$$iY_L(z) = g(z) + \frac{1}{L} g_b(z) + \frac{1}{L} \sum_{j=1}^{N} K(z_j, z)$$

- $z_1 = -\frac{1}{a} + \mathcal{O}(e^{-\mu_1 L})$ from pole of g_b

$$g_b(z) = \ldots + \ln\left[-\frac{1+az}{a+qz} \frac{1+cz}{c+qz} \right] + \ldots$$

- q-shifts $z_j = -\frac{q^{j-1}}{a} + \mathcal{O}(e^{-\mu_j L})$ from poles of K

$$K(w,z) = -\ln(w) - \ln\left(\frac{1-qz/w}{1-qw/z} \frac{1-q^2 wz}{1-wz} \right)$$

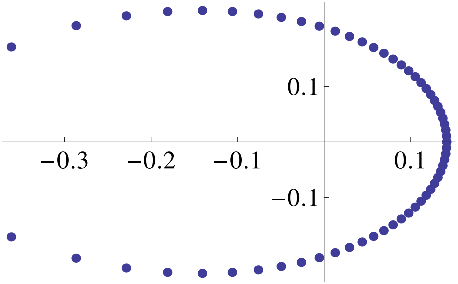
Sequence (S2)

- Define the sequence (S2)

$$\lambda_n^{(2)} = \ln\left(abcdq^{n-1}\right), \quad n = 0, 1, 2, \dots$$

satisfying second constraint with $k = L/2 - n$

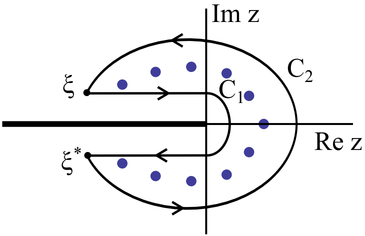
- $N = L - n$ Bethe roots



- Assume the form of the root distribution holds as $L \to \infty$

Integral equation for the counting function

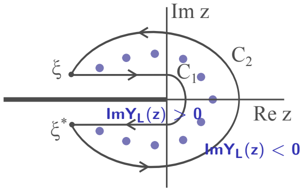
- Define contour of integration enclosing the Bethe roots



- Use the residue theorem to write the sum over roots as an integral

$$i\mathbf{Y_L}(z) = g(z) + \frac{1}{L}g_{\mathrm{b}}(z) + \oint_{C_1+C_2} \frac{dw}{4\pi i}\mathbf{K}(w,z)\mathbf{Y}'_{\mathbf{L}}(w)\cot\left(\frac{1}{2}L\mathbf{Y_L}(w)\right)$$

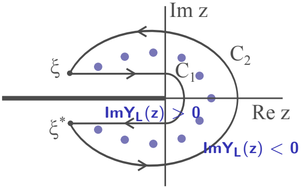
Saddle point approximation



- Rewrite integral equation, then approximate for L large

$$iY_L(z) = g(z) + \frac{1}{L} g_{\rm b}(z) + \int_{\xi^*}^{\xi} K(w,z) Y_L(w) dw + \int_{C_1} \frac{K(w,z) Y_L'(w)}{1 - e^{-iLY_L(w)}} dw + \int_{C_2} \frac{K(w,z) Y_L'(w)}{e^{iLY_L(w)} - 1} dw$$

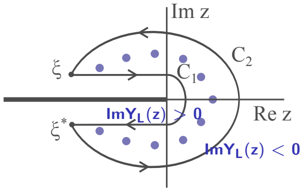
Saddle point approximation



- Rewrite integral equation, then approximate for L large

$$iY_L(z) = g(z) + \frac{1}{L}g_{\mathrm{b}}(z) + \int_{\xi^*}^{\xi} K(w,z)Y_L(w)dw$$
$$+ \int_{C_1} \frac{K(w,z)Y_L'(w)}{\mathbf{1 - e^{-iLY_L(w)}}}dw + \int_{C_2} \frac{K(w,z)Y_L'(w)}{\mathbf{e^{iLY_L(w)} - 1}}dw$$

Saddle point approximation



- Rewrite integral equation, then approximate for L large

$$
\begin{aligned}
iY_L(z) &= g(z) + \frac{1}{L} g_{\mathrm{b}}(z) + \int_{\xi^*}^{\xi} K(w,z) Y_L(w) dw \\
&\quad + \int_{C_1} \frac{K(w,z) Y_L'(w)}{\mathbf{1 - e^{-iLY_L(w)}}} dw + \int_{C_2} \frac{K(w,z) Y_L'(w)}{\mathbf{e^{iLY_L(w)} - 1}} dw \\
&\simeq g(z) + \frac{1}{L} g_{\mathrm{b}}(z) + \int_{\xi^*}^{\xi} K(w,z) Y_L(w) dw + \mathcal{O}(L^{-2})
\end{aligned}
$$

Asymptotic solution

Solve the integral equation:

- Find solution as asymptotic series in inverse powers of L

$$Y_L(z) = y_0(z) + \frac{1}{L} y_1(z) + \mathcal{O}(L^{-1})$$

- Similar analysis gives the asymptotic expression for the eigenvalue

$$E(\lambda_n^{(2)}) = (p - q) \frac{a(e^{\lambda_n^{(2)}} - 1)}{(1 + a)(e^{\lambda_n^{(2)}} + a)}$$

- Same form as (S1) result – conjecture it holds for all λ

$$E(\lambda) = (p - q) \frac{a(e^{\lambda} - 1)}{(1 + a)(e^{\lambda} + a)}$$

Results

- First moment reproduces the result for bulk current

$$\lim_{t\to\infty} \frac{\langle Q_1 \rangle}{t} = \frac{dE}{d\lambda}\bigg|_{\lambda=0} = (p-q)\rho(1-\rho)$$

where $\rho = 1/(1+a)$

- Second moment gives the current fluctuations, $q \to 0$ agrees with result for TASEP [Derrida, Evans, Mallick 1995]

$$\lim_{t\to\infty} \frac{\langle Q_1^2 \rangle - \langle Q_1 \rangle^2}{t} = \frac{dE^2}{d\lambda^2}\bigg|_{\lambda=0} = (p-q)\rho(1-\rho)(1-2\rho)$$

Relaxation rate

- Relaxation rate is the first excited eigenvalue

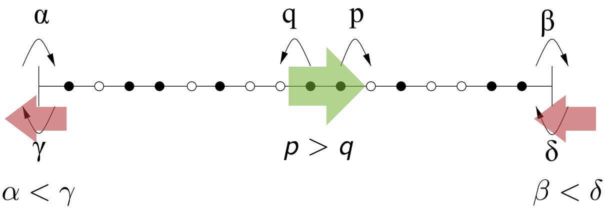
$$|P(t)\rangle \simeq a_0|\psi_0\rangle + a_1 e^{\mathcal{E}_1 t}|\psi_1\rangle$$

- Given by the **second set** of Bethe equations, with eigenvalue

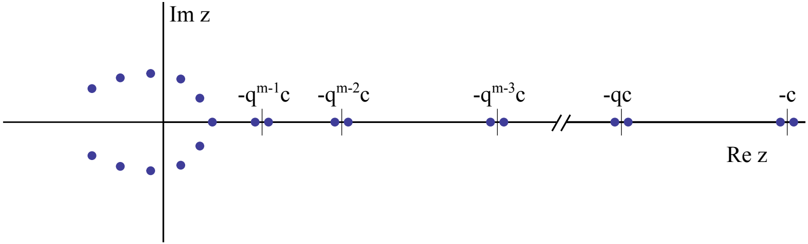
$$\mathcal{E} = -\mathcal{E}_0 + \sum_{j=1}^{L-1} \frac{(1-q)^2 z_j}{(1-z_j)(1-qz_j)}$$

Relaxation rate - the reverse bias regime

- Boundary rates oppose the bulk bias



The reverse bias regime – structure of solution

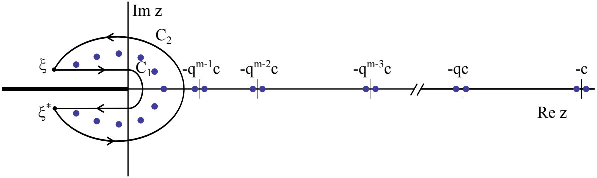


- m pairs of isolated roots

$$z_k^{\pm} = -q^k c \pm e^{-\nu_k^{\pm} L}, \quad k = 0, \ldots, m-1$$

- $L - 2m - 1$ roots z_j on the contour
- Analyse using combination of methods from (S1) and (S2)

Counting function for contour of roots



- Isolated roots telescope
- Work with scaled roots $\zeta_j = q^{-(m-1)} z_j$, dropping $\mathcal{O}(q^m)$ terms
- Bethe equations for the contour of roots

$$\overline{Y}_{L-2m}(\zeta_j) = \frac{2\pi}{L-2m} I_j, \quad j = 1, \dots, L-2m-1$$

- Counting function

$$i\overline{Y}_{L-2m}(\zeta) = \bar{g}(\zeta) + \frac{1}{L-2m}\bar{g}_{\mathrm{b}}(\zeta) + \frac{1}{L-2m}\sum_{l=1}^{L-2m-1}\overline{K}(\zeta_l, \zeta)$$

Asymptotic solution

Asymptotic solution for $L - 2m$ large

- Relaxation rate for the reverse bias regime

$$\mathcal{E}_1^{(\text{reverse})} = \frac{1-q}{(L-2m)^2}\frac{\pi^2}{(a^{-1}-a)} + \mathcal{O}((L-2m)^{-3})$$

- Compare forward bias regime [de Gier, Essler 2008]

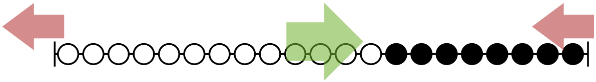
$$\mathcal{E}_1^{(\text{forward})} = \frac{1-q}{L^2}\frac{\pi^2}{(a^{-1}-a)} + \mathcal{O}(L^{-3})$$

Physical interpretation

- *Forward bias* vs *Reverse bias*

$$\frac{1-q}{L^2}\frac{\pi^2}{(a^{-1}-a)} \quad vs \quad \frac{1-q}{(L-2m)^2}\frac{\pi^2}{(a^{-1}-a)}$$

- Suggests the system fills completely over a length $2m$ from the right



- Behaves as in the forward bias regime with a reduced system size $L-2m$ (forms a uniformly distributed domain wall)
- Calculation of the stationary density profile required to confirm this picture

Conclusion

- Results come from detailed analysis of the Bethe root structure
- Derived expression characterising the asymptotic current distribution for the open ASEP
- Found the relaxation rate in the reverse bias regime differs from the forward bias regime
- To confirm the physical picture this suggests requires calculation of correlation functions for the ASEP, or equivalently the XXZ spin chain with non-diagonal open boundaries