Boundary bound states in spin-1 XXZ model and SUSY sine-Gordon model with Dirichlet boundaries

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Correlation Functions of Quantum Integrable Models
7-9 Sep 2011 Dijon (France)

Acknowledgement

I would like to acknowledge for
collaboration:

Francesco Buccheri (SISSA)

discussions:

Frank Göhmann (Wuppertal)
Andreas Klümper (Wuppertal)
Junji Suzuki (Shizuoka University)

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Spin-1 XXZ model with diagonal boundaries

Introduction

Various methods to calculate correlation functions:

- Vertex operators
- qKZ equations
- Bethe Ansatz

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Various methods to calculate correlation functions:

- Vertex operators
- qKZ equations
- Bethe Ansatz
  - We need to know the distribution of the Bethe roots.
  - ℓ-string solutions for the ground state of the periodic spin-$\ell/2$ XXZ chain ($\gamma < \pi/\ell$, in the thermodynamic limit.)
  - Under existence of boundary magnetic fields, boundary bound states appear (Jimbo et al. (94), Skorik&Saleur (95)), which changes the contour in the forms of correlation functions (Kitanine et al. (05).)

Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{M-1} H_{j,j+1} + \text{b.t.}$$

$$H_{j,j+1} = -T_j + (T_j)^2 + 2(\sin\gamma)^2 \left[T_j^z + (S_j^z)^2 + (S_{j+1}^z)^2 - (T_j^z)^2\right] - 4\left(\sin\frac{\gamma}{2}\right)^2 \left(T_j^\perp T_j^z + T_j^z T_j^\perp\right)$$

$$\text{b.t.} = \frac{1}{2}\sin 2\gamma \Bigg[\underbrace{-\left(\cot\frac{\gamma H_-}{2} + \cot\frac{\gamma}{2}(H_- + 2)\right)}_{h_1(H_-)} S_1^z \underbrace{+\left(\cot\frac{\gamma H_-}{2} - \cot\frac{\gamma}{2}(H_- + 2)\right)}_{h_2(H_-)} (S_1^z)^2 - \left(\cot\frac{\gamma H_+}{2} + \cot\frac{\gamma}{2}(H_+ + 2)\right) S_N^z + \left(\cot\frac{\gamma H_+}{2} - \cot\frac{\gamma}{2}(H_+ + 2)\right)(S_N^z)^2 \Bigg]$$

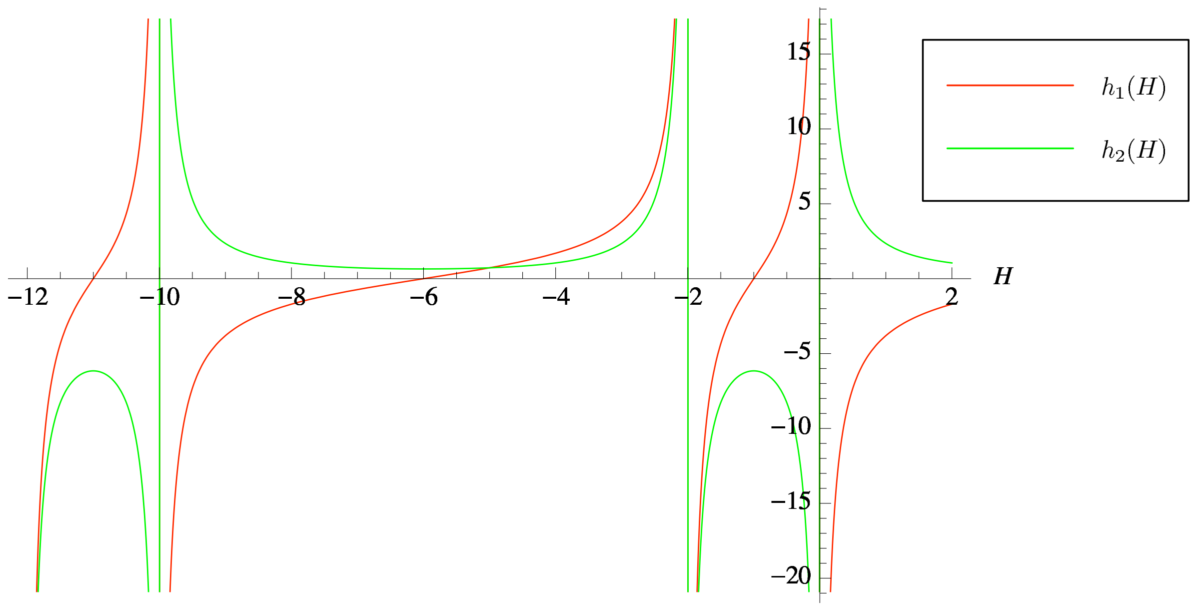
where

$$T_j = \vec{S}_j \cdot \vec{S}_{j+1} \qquad T_j^\perp = S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \qquad T_j^z = S_j^z S_{j+1}^z$$

$\mathbb{Z}_2$-symmetry

The Hamiltonian (2) has $\mathbb{Z}_2$-symmetry: $\chi\mathcal{H}\chi$; $\chi = \prod_{j=1}^{M} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{[j]}$

gives the same Hamiltonian but with $h_1(H) \to -h_1(H)$ and $h_2(H) \to h_2(H)$.



Since our vacuum is defined by $|1\dots1\rangle$, physically valid regimes are given by $h_1(H) > 0$.

Boundary bound states

- The conserved property of a set of momenta (assumption on the eigenfunctions)
- Integrable boundary conditions

lead to the Bethe Ansatz equations (diagonalization of the Hamiltonian)

$$\left[\frac{\sinh\frac{1}{2}(\alpha_j-2i\gamma)}{\sinh\frac{1}{2}(\alpha_j+2i\gamma)}\right]^{2M}\frac{\sinh\frac{1}{2}(\alpha_j-i\gamma H_-)\sinh\frac{1}{2}(\alpha_j-i\gamma H_+)}{\sinh\frac{1}{2}(\alpha_j+i\gamma H_-)\sinh\frac{1}{2}(\alpha_j+i\gamma H_+)}$$
$$\times\prod_{\substack{\ell=1\\ \ell\neq j}}^{m}\frac{\sinh\frac{1}{2}(\alpha_j-\alpha_\ell+2i\gamma)\sinh\frac{1}{2}(\alpha_j+\alpha_\ell+2i\gamma)}{\sinh\frac{1}{2}(\alpha_j-\alpha_\ell-2i\gamma)\sinh\frac{1}{2}(\alpha_j+\alpha_\ell-2i\gamma)}=1. \quad (1)$$

- On the half-infinite chain ($M \to \infty$, near the left ($-$) boundary), we obtain the pure imaginary solution $\alpha = -i\gamma H_-$ if $-t < H_- < 0$ ($t = \pi/\gamma$) from the asymptotic behavior of the first two terms of (1).
- Boundary string solutions are also obtained from the similar analysis as $\alpha = i\alpha_s^{\mathsf{B}} := -i\gamma H_- + 2i\gamma s$ ($s = -N, -N+1, \ldots, n$) where n and N are restricted by

$$\begin{aligned} &H_- > j - N && j = -1, \ldots, -N \\ &H_- < n + j && j = 1, \ldots, n \\ &H_- < n - N. \end{aligned}$$

Remark

The mirrored solutions $\alpha = -i\alpha_s^{\mathsf{B}}$ can be removed since they give diverging wave functions.

Energy shifts

Bethe Ansatz equations

$$\left[\frac{\sinh\frac{1}{2}(\alpha_j - 2i\gamma)}{\sinh\frac{1}{2}(\alpha_j + 2i\gamma)}\right]^{2M} \frac{\sinh\frac{1}{2}(\alpha_j - i\gamma H_-)\sinh\frac{1}{2}(\alpha_j - i\gamma H_+)}{\sinh\frac{1}{2}(\alpha_j + i\gamma H_-)\sinh\frac{1}{2}(\alpha_j + i\gamma H_+)}$$

$$= \prod_{\substack{\ell=1 \\ \ell\neq j}}^{m} \frac{\sinh\frac{1}{2}(\alpha_j - \alpha_\ell - 2i\gamma)\sinh\frac{1}{2}(\alpha_j + \alpha_\ell - 2i\gamma)}{\sinh\frac{1}{2}(\alpha_j - \alpha_\ell + 2i\gamma)\sinh\frac{1}{2}(\alpha_j + \alpha_\ell + 2i\gamma)}$$

- In the thermodynamic limit, most of the solutions for the ground state form the 2-strings: $\alpha = x \pm i\gamma$.
- Rewrite the BAE for the string centers: write down the BAE for each $\alpha = x \pm i\gamma$ at $m = M$ (M: an even number) and multiply them.

Taking logarithm of the BAE for string centers gives integral equations

$$f'(x,\gamma)+f'(x,3\gamma)$$
$$=\pi i\rho(x)+\frac{1}{2}\int_{-\infty}^{\infty}[2f'(x-y,2\gamma)+f'(x-y,4\gamma)]\rho(y)dy+\mathcal{O}(M^{-1})$$
$$f(\alpha,x)=\ln\left[\frac{\sinh\frac{1}{2}(\alpha-ix)}{\sinh\frac{1}{2}(\alpha+ix)}\right]$$
$$\rho(x)=\frac{1}{M(x_{j+1}-x_j)}.$$

cf. For a periodic chain

$$f'(x,\gamma)+f'(x,3\gamma)$$
$$=2\pi i\rho_{\mathrm{PBC}}(x)+\int_{-\infty}^{\infty}[2f'(x-y,2\gamma)+f'(x-y,4\gamma)]\rho_{\mathrm{PBC}}(y)dy$$

The density $\rho(\alpha)$ is twice the one for a periodic chain

$$\rho(x)=2\rho_{\mathrm{PBC}}(x)+\mathcal{O}(M^{-1}).$$

Remark

$\rho(x)$ has $\mathcal{O}(M^{-1})$ correction from $2\rho_{\mathrm{PBC}}(\alpha)$:

$$
\begin{aligned}
&f'(x,\gamma) + f'(x,3\gamma) \\
&= \pi i \rho_{n,N}(x) + \frac{1}{2}\int_{-\infty}^{\infty}[2f'(x-y,2\gamma) + f'(x-y,4\gamma)]\rho_{n,N}(y)dy + \mathcal{O}(M^{-1})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}(M^{-1}) = \frac{1}{2M}\{&2f'(2x,2\gamma) + f'(2x,4\gamma) \\
&- f'(x,\gamma H_- + \gamma) - f'(x,\gamma H_- - \gamma) - \cancel{(H_- \to H_+)} \\
&+ \underbrace{\sum_{s=-N}^{n}[f'(x - i\alpha_s^{\mathrm{B}},\gamma) + f'(x - i\alpha_s^{\mathrm{B}},3\gamma) + (- \to +)]}_{\text{if bbs } i\alpha_s^{\mathrm{B}} = -i\gamma H + 2i\gamma s \text{ exist}}\}.
\end{aligned}
$$

Since what we want to know is energy shifts due to bbs, we define density shift $\delta\rho_{n,N}(\alpha)$ by

$$\delta\rho_{n,N}(x) = 2M[\rho_{n,N}(x) - \rho_0(x)].$$

$\rho_{n,N}(x)$: Bethe root density with boundary (n,N)-string solutions

$\rho_0(x)$: Bethe root density without boundary bound solutions

$\delta\rho_{n,N}(\alpha)$ satisfies an integral equation:

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} [f'(x-y, 4\gamma) + 2f'(x-y, 2\gamma)]\delta\rho_{n,N}(y)dy \\
&\quad + \psi'_{\text{bdry}}(x) + 4\pi i \delta\rho_{n,N}(x) \\
\psi'_{\text{bdry}}(x) &= \sum_{s=-N}^{n} [f'(x - i\alpha_s^{\text{B}}, \gamma) + f'(x + i\alpha_s^{\text{B}}, \gamma) \\
&\quad + f'(x - i\alpha_s^{\text{B}}, 3\gamma) + f'(x + i\alpha_s^{\text{B}}, 3\gamma)
\end{aligned}
\tag{2}
$$

(2) can be solved by Fourier transformation. The expression of the Fourier transformation depends on the value of H_-: we set $0 < H_- < 1$ in order to compare the result with SSG case:

$$
\begin{cases}
\dfrac{\cosh(\alpha_0^{\mathsf{B}}+2\gamma s-\pi)k \sinh 2\gamma k \cosh \gamma k}{\sinh(\pi-2\gamma)k \cosh^2 \gamma k} & s \geq 2 \\[2ex]
\dfrac{\cosh(\alpha_0^{\mathsf{B}}+2\gamma s-\pi)k \sinh \gamma k-\cosh(\alpha_0^{\mathsf{B}}+2\gamma s)k \sinh(\pi-3\gamma)k}{2\sinh(\pi-2\gamma)k \cosh^2 \gamma k} & s = 1 \\[2ex]
\dfrac{-\cosh(\alpha_0^{\mathsf{B}}+2\gamma s)k \sinh(\pi-2\gamma)k \cosh \gamma k}{\sinh(\pi-2\gamma)k \cosh^2 \gamma k} & s = 0 \\[2ex]
\dfrac{\cosh(\alpha_0^{\mathsf{B}}+2\gamma s+\pi)k \sinh \gamma k-\cosh(\alpha_0^{\mathsf{B}}+2\gamma s)k \sinh(\pi-3\gamma)k}{2\sinh(\pi-2\gamma)k \cosh^2 \gamma k} & s = -1 \\[2ex]
\dfrac{\cosh(\alpha_0^{\mathsf{B}}+2\gamma s+\pi)k \sinh 2\gamma k \cosh \gamma k}{\sinh(\pi-2\gamma)k \cosh^2 \gamma k} & s \leq -2
\end{cases}
\tag{3}
$$

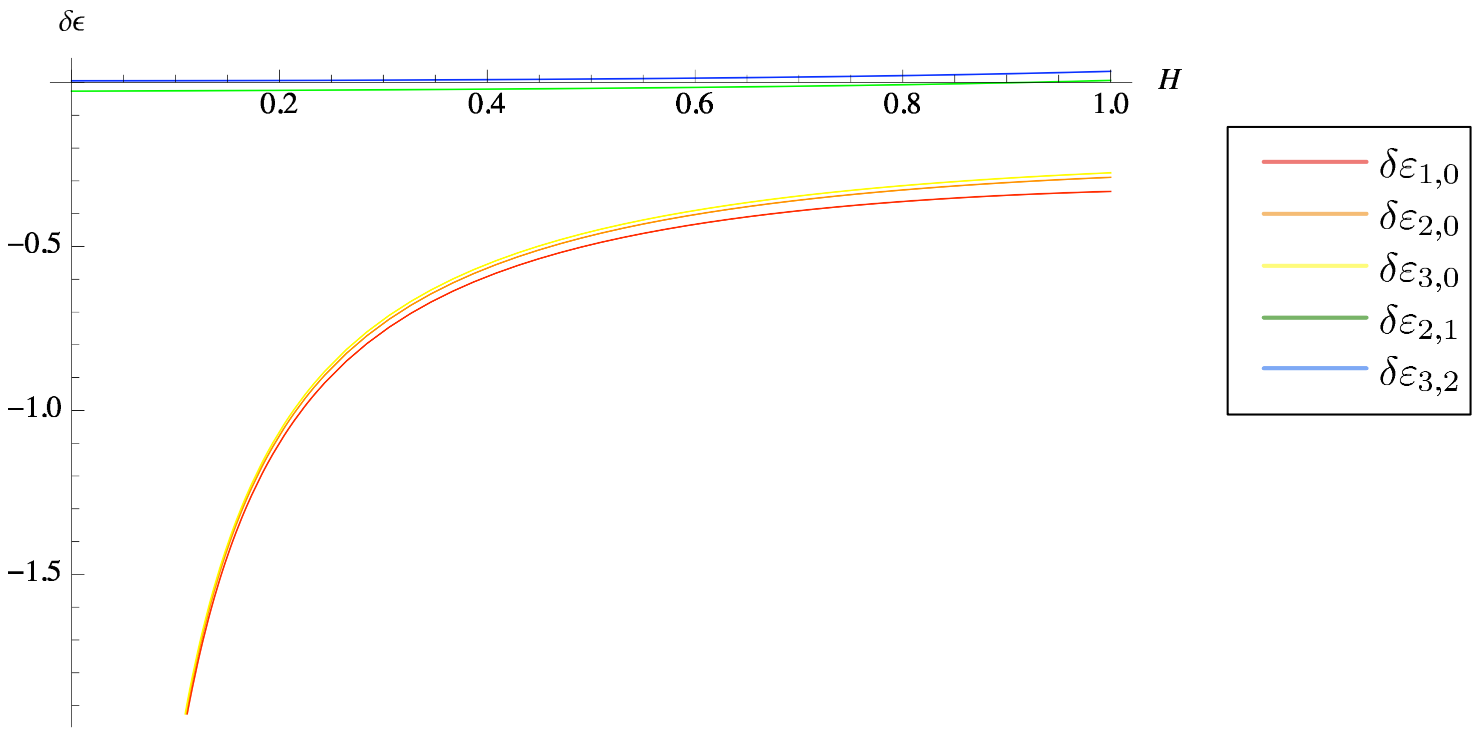
Eigenenergy is given by

$$E = \frac{i\gamma}{8\pi} \sum_{j} f'(\alpha_j, 2\gamma). \tag{4}$$

In the thermodynamic limit, density of the 2-string centers becomes dense and a summation in (4) is written as an integral and boundary parts. Thus energy shift $\delta E_{n,N}$ is given by

$$\begin{aligned} \delta E_{n,N} = & \frac{i\gamma}{8\pi} \int_{-\infty}^{\infty} [f'(x+i\gamma, 2\gamma) + f'(x-i\gamma, 2\gamma)] \delta\rho_{n,N}(x) dx \\ & + \sum_{s=-N}^{n} \frac{i\gamma}{4\pi} f'(i\alpha_s^{\mathsf{B}}, 2\gamma). \end{aligned} \tag{5}$$

Plots for all the string configurations allowed in $0 < H_- < 1$:



Conclusion

- *The 2-string solutions give the lowest energy.*
- $\alpha = i\alpha_0^{\mathsf{B}}$ *and* $i\alpha_1^{\mathsf{B}}$ *contribute to the ground state.*

SUSY sine-Gordon model with Dirichlet boundaries

Lagrangian

$$\mathcal{L}_{\mathsf{SSG}} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - \mathsf{m}\bar{\Psi}\Psi \cos\frac{\beta\Phi}{2} - \frac{\mathsf{m}^2}{\beta^2}\cos\beta\Phi$$

Φ: a scalar boson $\qquad$ Ψ: a Majorana fermion

Impose Dirichlet boundary conditions on the fields:

$$\Phi(0,t) = \Phi_+ \qquad \Psi(0,t) = \bar{\Psi}(0,t)$$
$$\Phi(L,t) = \Phi_- \qquad \Psi(L,t) = \bar{\Psi}(L,t)$$

SSG model $\underset{\text{Lattice regularization}}{\Longleftrightarrow}$ spin-1 XXZ model

Lattice regularization

The lattice regularized SUSY sine-Gordon model is described by the spin-1 XXZ model with the alternating inhomogeneity $\pm\Lambda$ in the scaling limit:

- the inhomogemeity $\Lambda \to \infty$
- the system size $M \to \infty$
- the lattice spacing $a \to 0$

such that

- the soliton mass $\mathrm{m} = 2M \exp\left(-\frac{\pi}{2\gamma}\Lambda\right)$
- the system length $L = Ma$

Boundary S-matrix (Ahn et al. (91,06))

The boundary S-matrix of the BSSG model consists of two parts:

$$\mathcal{R}(\theta;\xi_\pm) := \mathcal{R}_{\mathsf{RSOS}}(\theta) \times \mathcal{R}_{\mathsf{SG}}(\theta;\xi_\pm)$$

which has an integral expression:

$$\mathcal{R}_{\mathsf{RSOS}}(\theta) \sim \exp\left(\frac{i}{8}\int_0^\infty \frac{dt}{t}\,\frac{\sin(2t\theta/\pi)}{\cosh^2\frac{t}{2}\cosh^2 t}\right) \tag{6}$$

$$\frac{1}{i}\frac{d}{d\theta}\ln\mathcal{R}_{\mathsf{SG}}(\theta;\xi_\pm) = \int_{-\infty}^{\infty} dke^{-ik\theta}\left[\frac{\sinh((1+\frac{2\xi_\pm}{\pi\lambda})\frac{\pi k}{2})}{2\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{2\lambda}} + \frac{\sinh\frac{3\pi k}{4}\sinh((\frac{1}{\lambda}-1)\frac{\pi k}{2})}{\sinh\pi k\sinh\frac{\pi k}{4\lambda}}\right] \tag{7}$$

$$= \int_{-\infty}^{\infty} dke^{-ik\theta}\mathrm{sign}(H_\pm)\frac{\sinh\frac{\pi}{2}(t-|H_\pm|)k}{2\cosh\frac{\pi}{2}k\sinh\frac{\pi}{2}(t-2)k} + \int_{-\infty}^{\infty} dke^{-ik\theta}\frac{\sinh\frac{3\pi}{4}k\sinh\frac{\pi}{2}(t-3)k}{\sinh\pi k\sinh\frac{\pi}{4}(t-2)k}. \tag{8}$$

In the context of the spin chain, the boundary S-matrix corresponds to the $\mathcal{O}(M^{-1})$ corrections in the nonlinear integral equations of the auxiliary functions, which is derived from analyticity of the two valid transfer matrices $T_1(\theta)$ and $T_2(\theta)$ under the conditions:

$$0 < H_\pm < t$$
$$-2 < H_+ + H_- < \frac{8}{3}t - 2.$$

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- Dirac sea of two strings ⇔ the vacuum of SSG model

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$$
\begin{aligned}
0 <& H_\pm < t \\
-2 < H_+ + H_- <& \frac{8}{3}t - 2.
\end{aligned}
$$

- Dirac sea of two strings $\Leftrightarrow$ the vacuum of SSG model
- a hole at $\theta = \theta_h$ in the distribution of two strings (a real zero of $T_2(\theta)$) $\Leftrightarrow$ a SSG soliton with rapidity θ_h

Nonlinear integral equations

For a state of one hole θ_h:

$$\ln \mathsf{b}(\theta) = \int_{-\infty}^{\infty} d\theta' \mathsf{G}(\theta - \theta' - i\epsilon) \ln \mathsf{B}(\theta' + i\epsilon)$$
$$+ \int_{-\infty}^{\infty} d\theta' \mathsf{G}(\theta - \theta' + i\epsilon) \ln \bar{\mathsf{B}}(\theta' - i\epsilon)$$
$$+ \int_{-\infty}^{\infty} d\theta' \mathsf{G}_2(\theta - \theta' - i\epsilon) \ln \mathsf{Y}(\theta' + i\epsilon)$$
$$+ i2\mathsf{m}L \sinh\theta + i\mathsf{P}_{\mathsf{bdry}}(\theta) + ig(\theta - \theta_h) + ig(\theta + \theta_h) - i\pi$$
$$\ln \mathsf{y}(\theta) = \int_{-\infty}^{\infty} d\theta' \mathsf{G}_2(\theta - \theta' + i\epsilon) \ln \bar{\mathsf{B}}(\theta' - i\epsilon)$$
$$+ \int_{-\infty}^{\infty} d\theta' \mathsf{G}_2(\theta - \theta' + i\epsilon) \ln \mathsf{B}(\theta' + i\epsilon)$$
$$+ i\mathsf{P}_y(\theta) + ig_y(\theta - \theta_h) + ig_y(\theta + \theta_h)$$

- $T_2(\theta_h) = 0 \Leftrightarrow 1 + \mathsf{b}(\theta_h) = 0$
- Analysis in the limit $\mathsf{m}L \to \infty$ gives the information of the boundary S-matrix:
  - $\ln \mathsf{B}$, $\ln \bar{\mathsf{B}} \to 0$

$$\exp[i2\mathsf{m}L \sinh \theta_h + i\mathsf{P}_{\mathsf{bdry}}(\theta_h) + ig(2\theta_h) + \mathcal{K}(\theta_h)] = 1 \tag{9}$$

c.f. Yang equation

$$\exp(i2\mathsf{m}L \sinh \theta_h)\mathcal{R}(\theta_h; \lambda, \xi_-)\mathcal{R}(\theta_h; \lambda, \xi_+) = 1 \tag{10}$$

Comparing (9) with (10) we have

$$\mathcal{R}(\theta_h; \lambda, \xi_-)\mathcal{R}(\theta_h; \lambda, \xi_+) = \exp[\underbrace{i\mathsf{P}_{\mathsf{bdry}}(\theta_h) + ig(2\theta_h)}_{\text{the SG factor}} + \underbrace{\mathcal{K}(\theta_h)}_{\text{the RSOS factor}}].$$

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$$\begin{aligned}\frac{1}{i}\frac{d}{d\theta}\ln\mathcal{R}_{\mathsf{SG}}(\theta;\xi_\pm) = \int_{-\infty}^{\infty} dk e^{-ik\theta}\Bigg[&\frac{\sinh((1+\frac{2\xi_\pm}{\pi\lambda})\frac{\pi k}{2})}{2\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{2\lambda}} \\ &+ \frac{\sinh\frac{3\pi k}{4}\sinh((\frac{1}{\lambda}-1)\frac{\pi k}{2})}{\sinh\pi k\sinh\frac{\pi k}{4\lambda}}\Bigg]\end{aligned} \tag{12}$$

$$\begin{aligned}&= \int_{-\infty}^{\infty} dk e^{-ik\theta}\mathsf{sign}(H_\pm)\frac{\sinh\frac{\pi}{2}(t-|H_\pm|)k}{2\cosh\frac{\pi}{2}k\sinh\frac{\pi}{2}(t-2)k} \\ &+ \int_{-\infty}^{\infty} dk e^{-ik\theta}\frac{\sinh\frac{3\pi}{4}k\sinh\frac{\pi}{2}(t-3)k}{\sinh\pi k\sinh\frac{\pi}{4}(t-2)k}.\end{aligned} \tag{13}$$

Boundary bound states

Two expressions of the boundary S-matrix (12) and (13) give the parameter relations

$$t = p + 2 \qquad t - H_{\pm} = 1 + \frac{2\xi_{\pm}}{\pi} p.$$

- Boundary bound states appear when $\xi_{\pm} > \frac{\pi}{2} \Leftrightarrow H_{\pm} < 1$.
- Boundary dependence comes from the first term of (13).
- Poles give rapidity of boundary bound solitons:

$$\mathcal{R}e(\theta) = 0 \qquad \mathcal{I}m(\theta) = \pm\frac{\pi}{2}(1 - H_{-} + 2pn) \qquad n \in \mathbb{Z}$$

c.f. Poles in the boundary SG S-matrix:

$$\nu_n = \xi_{-} p - \left(n + \frac{1}{2}\right) \pi p$$

These poles are on the physical strip $\Rightarrow n$ is restricted by

$$0 < \frac{\pi}{2}(1 - H_- + 2pn) < \frac{\pi}{2}.$$

Thus for $0 < H_- < 1$, we have $n = 0$.

Remark

The pole $\theta = \frac{i\pi}{2}(1 - H_-)$ is interpreted as a string center of two boundary string solutions $\theta := \frac{\pi}{2\gamma}\alpha = -\frac{i\pi H_-}{2}$ and $-\frac{\pi}{2}(H_- - 2)$ of the BAE.

Boundary bound states

The mass of the one hole state $\theta = \frac{i\pi}{2}(1 - H_-)$ is calculated in the scaling limit:

- $\Lambda \to \infty$
- Choose the pole at $k = \pi i/2\gamma$

$$
\begin{aligned}
\mathrm{m}_{1,0} &= \frac{i\gamma}{4\pi a} \sum_{s=0}^{1} [f'(i\alpha_s^{\mathrm{B}} - \Lambda, 2\gamma) + f'(i\alpha_s^{\mathrm{B}} + \Lambda, 2\gamma)] \\
&+ \frac{i\gamma}{8\pi a} \int_{-\infty}^{\infty} \delta\rho_{1,0}(x)[f'(x + i\gamma - \Lambda, 2\gamma) + f'(x + i\gamma + \Lambda, 2\gamma) \\
&+ f'(x - i\gamma - \Lambda, 2\gamma) + f'(x - i\gamma + \Lambda, 2\gamma)]dx \\
&\to -\frac{\mathrm{m}}{2} \sin\left(\frac{\pi H_-}{2}\right) \le 0
\end{aligned}
$$

Conclusion

The ground state consisting of 2-strings is unstable. Thus the "correct" ground state contains the boundary bound states.

Summary

- For the lattice model,
  - Restriction on the configuration of boundary bound states from the value of H_-
  - Derivation of energy shifts due to boundary bound states
- For the BSSG model,
  - Deriving rapidity of boundary bound states
  - Stability of the testing ground state
- Both models have the ground states consists of the bulk 2-strings and the boundary 2-string solutions in $0 < H_- < 1$.
- Calculating correlation functions
- How about in the ferromagnetic regime?

Appendix: Auxiliary functions

Auxiliary functions

The auxiliary functions consist of three functions

$$\mathsf{b}(x) = \frac{\lambda_1(x) + \lambda_2(x)}{\lambda_3(x)} \qquad \bar{\mathsf{b}}(x) = \mathsf{b}(-x)$$

$$\mathsf{y}(x) = \frac{T_0(x) T_2(x)}{f(x)}$$

$$T_0(x) = \sinh(2x)$$

$$f(x) = l_2\left(x - \frac{i\gamma}{2}\right) l_1\left(x + \frac{i\gamma}{2}\right)$$

and functions related to each of them

$$\mathsf{B}(x) = 1 + \mathsf{b}(x) \qquad \bar{\mathsf{B}}(x) = 1 + \bar{\mathsf{b}}(x) \qquad \mathsf{Y}(x) = 1 + \mathsf{y}(x).$$

Each function appeared in the auxiliary functions comes from the transfer matrices

$$T_1(x) = l_1(x) + l_2(x)$$
$$T_2(x) = \lambda_1(x) + \lambda_2(x) + \lambda_3(x)$$

with explicit forms

$$l_1(x) = \sinh(2x + i\gamma)B_+(x)\phi(x + i\gamma)\frac{Q(x - i\gamma)}{Q(x)}$$
$$l_2(x) = \sinh(2x - i\gamma)B_-(x)\phi(x - i\gamma)\frac{Q(x + i\gamma)}{Q(x)}$$

$$\lambda_1(x) = \sinh(2x - 2i\gamma) B_- \left(x - \frac{i\gamma}{2}\right) B_- \left(x + \frac{i\gamma}{2}\right)$$
$$\times \phi\left(x - \frac{3i\gamma}{2}\right) \phi\left(x - \frac{i\gamma}{2}\right) \frac{Q(x + \frac{3i\gamma}{2})}{Q(x - \frac{i\gamma}{2})}$$
$$\lambda_2(x) = \sinh(2x) B_+ \left(x - \frac{i\gamma}{2}\right) B_- \left(x + \frac{i\gamma}{2}\right)$$
$$\times \phi\left(x - \frac{i\gamma}{2}\right) \phi\left(x + \frac{i\gamma}{2}\right) \frac{Q(x + \frac{3i\gamma}{2}) Q(x - \frac{3i\gamma}{2})}{Q(x - \frac{i\gamma}{2}) Q(x + \frac{i\gamma}{2})}$$
$$\lambda_3(x) = \sinh(2x + 2i\gamma) B_+ \left(x - \frac{i\gamma}{2}\right) B_+ \left(x + \frac{i\gamma}{2}\right)$$
$$\times \phi\left(x + \frac{3i\gamma}{2}\right) \phi\left(x + \frac{i\gamma}{2}\right) \frac{Q(x - \frac{3i\gamma}{2})}{Q(x + \frac{i\gamma}{2})}$$

with functions

$$\phi(x) = \sinh^M(x - \Lambda) \sinh^M(x + \Lambda)$$

$$B_\pm(x) = \sinh\left(x \pm \frac{i\gamma H_-}{2}\right) \sinh\left(x \pm \frac{i\gamma H_+}{2}\right)$$

$$Q(x) = \prod_{k=1}^{m} \sinh(x - x_k) \sinh(x + x_k)$$

where x_k as Bethe roots.

Appendix: Definition of functions

$$\hat{\mathsf{G}}(k) = \frac{\sinh(\pi - 3\gamma)\frac{k}{2}}{2\cosh\frac{\gamma k}{2}\sinh((\pi-2\gamma)\frac{k}{2})} \qquad g(\theta) = 2\pi\int_0^\theta d\theta' \mathsf{G}(\theta')$$

$$\hat{\mathsf{G}}_2(k) = \frac{e^{-\frac{\gamma k}{2}}}{e^{\frac{\gamma k}{2}} + e^{-\frac{\gamma k}{2}}} \qquad g_y(\theta) = -i\ln\tanh\frac{\theta}{2} + \frac{\pi}{2}$$

$$\mathsf{P}_{\mathsf{bdry}}(\theta) = \frac{\gamma}{4\pi^2}\int_{-\theta}^{\theta} d\theta' \int_{-\infty}^{\infty} dk e^{-ik\gamma\theta'/\pi}\hat{R}(k)$$

$$\hat{R}(k) = 2\pi\left[\hat{F}(k, H_-) + \hat{F}(k, H_+) + \frac{\cosh\frac{\gamma k}{4}\sinh(3\gamma-\pi)\frac{k}{4}}{\cosh\frac{\gamma k}{2}\sinh(2\gamma-\pi)\frac{k}{4}}\right]$$

$$\hat{F}(k, H) = \mathsf{sign}(H)\frac{\sinh(\pi - \gamma|H|)\frac{k}{2}}{2\cosh\frac{\gamma k}{2}\sinh(\pi-2\gamma)\frac{k}{2}}$$

$$\mathsf{P}_y(\theta) = -2i\ln\tanh\frac{\theta}{2} - 2\pi$$