Some properties of functions related to the six-vertex model with disorder parameter

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Outline

- Generalized density matrix and correlation functions
- Algebraic structure
- Analytic structure: ρ and ω
- Asymptotics and functional equations
- Universal finite size corrections in the isotropic case

Hamiltonian and spectrum

- Anisotropic Heisenberg chain

$$\mathcal{H}_L = J \sum_{j=-L+1}^{L} \left(\sigma^x_{j-1}\sigma^x_j + \sigma^y_{j-1}\sigma^y_j + \Delta \sigma^z_{j-1}\sigma^z_j \right)$$

$\Delta = \mathrm{ch}(\eta) = (q + q^{-1})/2$

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(1) free energy per lattice site

$$f(T,h) = -T \lim_{L\to\infty} \frac{1}{L} \ln \mathrm{tr} \exp\left\{ -\frac{\mathcal{H}_L}{T} + \frac{h\mathcal{S}^z_L}{T} \right\}$$

$\rightarrow$ TD, one-point functions, CFT from low T

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- (2) Ground state energy of finite system

$$E(L) = \lim_{T\to 0} T^2 \partial_T \ln \mathrm{tr} \exp\Big\{ -\frac{\mathcal{H}_L}{T} \Big\}$$

$\rightarrow$ finite-size corrections, CFT

Density matrix and correlation functions

- Measurable quantities related to the eigenvectors:

$$\langle \mathcal{O} \rangle_{T,h} = \lim_{L\to\infty} \mathrm{tr}\, \rho_L \mathcal{O}\,, \qquad \rho_L = \frac{\exp\left\{-\frac{\mathcal{H}_L}{T} + \frac{h\mathcal{S}_L^z}{T}\right\}}{\mathrm{tr}\exp\left\{-\frac{\mathcal{H}_L}{T} + \frac{h\mathcal{S}_L^z}{T}\right\}}$$

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$$D_{[1,n]}(T,h) = \lim_{L \to \infty} \operatorname{tr}_{-L+1,\dots,0,n+1,\dots,L} \rho_L$$

  (reduced) density matrix

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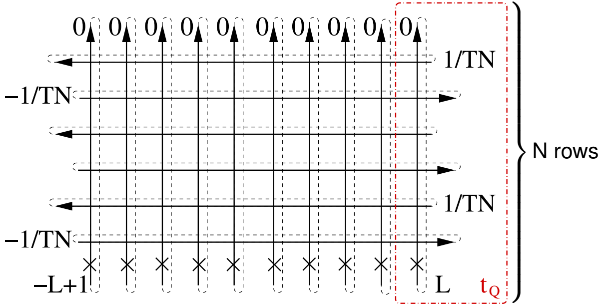
- Then, for operators of finite length,

$$\langle \mathcal{O} \rangle_{T,h} = \lim_{n\to\infty} \operatorname{tr}_{1,\dots,n} D_{[1,n]}(T,h) \mathcal{O}_{[1,n]}$$

  „inductive limit“

Partition function

$$Z_L = \lim_{N\to\infty}$$



$$\bar{t}(1/NT)t(-1/NT) = 1 - \frac{2\mathcal{H}_L}{NT} + \mathcal{O}(1/N^2)$$

$$\lim_{N\to\infty} \left(\bar{t}(1/NT)t(-1/NT)\right)^{\frac{N}{2}} = \mathrm{e}^{-\mathcal{H}_L/T}$$

$$= \left(q^{\kappa\sigma^z}\right)^\alpha_\beta$$

t_Q quantum transfer matrix, dominant eigenvalue $\Lambda(\lambda|\kappa)$

Eigenvalue and integral equation

The dominant eigenvalue can be represented as

$$\Lambda(\lambda|\kappa) = \kappa\eta + \int_{\mathcal{C}} \frac{d\mu}{2\pi i} e(\mu - \lambda) \ln\bigl(1 + \mathfrak{a}(\mu, \kappa)\bigr)$$

with the bare energy $e(\lambda) = \mathrm{cth}(\lambda) - \mathrm{cth}(\lambda + \eta)$,

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$$\ln(\mathfrak{a}(\lambda,\kappa)) = -2\kappa\eta - \frac{2J\,\mathrm{sh}(\eta)\mathrm{e}(\lambda)}{T} - \int_{\mathcal{C}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} K_0(\lambda-\mu) \ln(1+\mathfrak{a}(\mu,\kappa))$$

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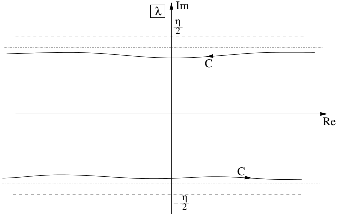
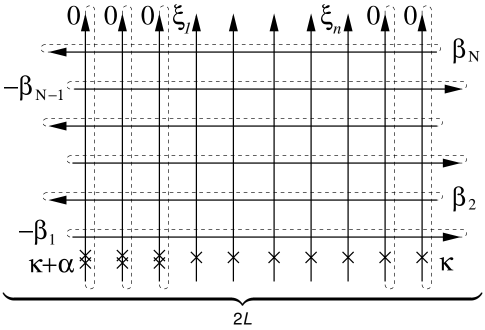
and kernel and integration contour

$$K_\alpha(\lambda) = q^{-\alpha}\,\mathrm{cth}(\lambda-\eta) - q^{\alpha}\,\mathrm{cth}(\lambda+\eta)$$



Generalized density matrix and reduction



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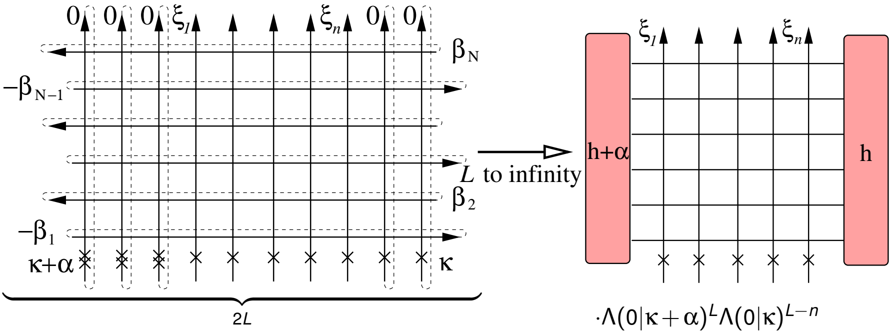

$$\cdot\Lambda(0|\kappa+\alpha)^{L}\Lambda(0|\kappa)^{L-n}$$

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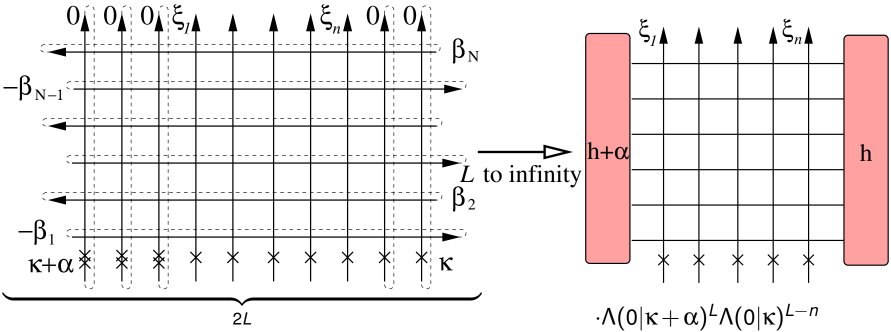

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$$\boxed{\to D_{[1,n]}(\xi_1,\dots,\xi_n|T,\kappa,\alpha,N)}$$

$\sum\beta_j = 1/T$, generalized density matrix

- reduction from left and right

$$\mathrm{tr}_1\left\{D_{[1,m]}(\xi_1,\dots,\xi_m|\kappa,\alpha)q^{\alpha\sigma_1^z}\right\} = \rho(\xi_1)D_{[2,m]}(\xi_2,\dots,\xi_m|\kappa,\alpha)$$

$$\mathrm{tr}_m\left\{D_{[1,m]}(\xi_1,\dots,\xi_m|\kappa,\alpha)\right\} = D_{[1,m-1]}(\xi_1,\dots,\xi_{m-1}|\kappa,\alpha)$$

- where

$$\rho(\lambda) = \frac{\Lambda(\lambda|\kappa+\alpha)}{\Lambda(\lambda|\kappa)}$$

Space of quasi-local operators (BJMST 07-09)

- Reduction determines one-point functions

$$\begin{pmatrix} D_+^+(\xi) \\ D_-^-(\xi) \end{pmatrix} = \frac{1}{q^\alpha - q^{-\alpha}} \begin{pmatrix} \rho(\xi) - q^{-\alpha} \\ q^\alpha - \rho(\xi) \end{pmatrix}$$

- Physical correlation functions for $\alpha \to 0$,

$$\rho(1) = 1 + m(T,h) 2\eta\alpha + \mathcal{O}(\alpha^2)$$

with $h = 2\kappa\eta T$ and with the magnetization $m(T,h)$

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- Define $S(k) = \frac{1}{2}\sum_{-\infty}^{k} \sigma_k^z$. If $\mathcal{O}$ is local, we call $q^{2\alpha S(0)}\mathcal{O}$ quasi-local. The vector space of all such operators is denoted $\mathcal{W}_\alpha$, the restriction of $\mathcal{O}$ onto a segment $[k,\ell]$ is denoted $\mathcal{O}_{[k,\ell]}$
- For a lattice of infinite extension in horizontal direction we define the functional $Z^\kappa : \mathcal{W}_\alpha \to \mathbb{C}$ as an inductive limit

$$Z^\kappa(X) =$$

$$\lim_{\ell\to\infty} \mathrm{tr}_{-\ell+1,\dots,\ell}\big(\rho^{-\ell}(1) D_{[-\ell+1,\ell]}(\kappa,\alpha) X_{[-\ell+1,\ell]}\big)$$

For $X = q^{2\alpha S(k-1)} X_{[k,m]}$ we have

$$Z^\kappa(X) = \rho^{k-1}(1) \big\langle X_{[k,m]} \big\rangle_\kappa$$

Z^κ can be interpreted as the statistical operator on $\mathcal{W}_\alpha$

Fermions on the space of quasi-local operators

- We say that $X \in \mathcal{W}_\alpha$ has spin s if $[S(\infty), X] = sX$. For $s \in \mathbb{Z}$ denote the spin-s subspace of $\mathcal{W}_\alpha$ as $\mathcal{W}_{\alpha,s}$. Consider

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}$$

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- On $\mathcal{W}^{(\alpha)}$ exist special Fermi operators with mode expansion (BJMST 07, 08)

$$\mathbf{b}(\zeta) = \zeta^{-\alpha} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{b}_p$$

$$\mathbf{b}^*(\zeta) = \zeta^{\alpha+2} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{b}^*_p$$

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- They mutually anti-commute, except

$$\{\mathbf{b}(\zeta_1), \mathbf{b}^*(\zeta_2)\} = -\psi(\zeta_2/\zeta_1)$$
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$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}^*_p$$

exists which commutes with the Fermions. Its first Fourier mode $\mathbf{t}^*_1$ is (twice) the shift operator

Fermions on the space of quasi-local operators

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- All Fourier modes have block structure

$$\mathbf{t}^*_p : \mathcal{W}_{\alpha-s,s} \to \mathcal{W}_{\alpha-s,s}$$
$$\mathbf{b}^*_p, \mathbf{c}_p : \mathcal{W}_{\alpha-s+1,s-1} \to \mathcal{W}_{\alpha-s,s}$$
$$\mathbf{c}^*_p, \mathbf{b}_p : \mathcal{W}_{\alpha-s-1,s+1} \to \mathcal{W}_{\alpha-s,s}$$

Two fundamental theorems

- THEOREM (BJMS 09): The Fourier modes

$$\tau^m \mathbf{t}^*_{p_1} \dots \mathbf{t}^*_{p_j} \mathbf{b}^*_{q_1} \dots \mathbf{b}^*_{q_k} \mathbf{c}^*_{r_1} \dots \mathbf{c}^*_{r_k} (q^{2\alpha S(0)})$$

$m \in \mathbb{Z}$, $j, k \in \mathbb{N}$, $p_1 \geq p_2 \geq \dots \geq p_j$, $q_1 > q_2 > \dots > q_k$ and $r_1 > r_2 > \dots > r_k$ generate a basis of $\mathcal{W}_{\alpha,0}$ (over the Fock vacuum $q^{2\alpha S(0)}$)

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- COROLLARY: Factorization lemma

$$Z^\kappa\{\mathbf{t}^*(\zeta_1^0)\dots\mathbf{t}^*(\zeta_k^0)\mathbf{b}^*(\zeta_1^+)\dots\mathbf{b}^*(\zeta_\ell^+) \mathbf{c}^*(\zeta_1^-)\dots\mathbf{c}^*(\zeta_\ell^-) q^{2\alpha S(0)}\} = \det_{i,j=1,\dots,\ell}[\omega(\zeta_i^+,\zeta_j^-)] \prod_{p=1}^k 2\rho(\zeta_p^0)$$

Taylor expansion of both sides in $(\zeta_i^\varepsilon)^2 - 1$ yields Z^κ on every basis element! It follows that any correlation function is a polynomial in ρ, ω and their derivatives

The function ω – characterization by integral equations

- THEOREM (BG 09): The function ω can be fully described by means of solutions of linear and non-linear integral equations.

$$e^{\alpha(\nu_2-\nu_1)}\omega(\nu_1,\nu_2|\kappa,\alpha) = 2\Psi^t(\nu_1,\nu_2) + \frac{1}{2}K_\alpha(\nu_1-\nu_2) + \big(\rho(\nu_1)-\rho(\nu_2)\big)\,\mathrm{cth}(\nu_1-\nu_2)$$

where (for ν_1 inside $\mathcal{C}$)

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- Here we introduced ‘the Fermi measure’

$$\mathrm{d}m(\lambda) = \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}\,\rho(\lambda)(1+\mathfrak{a}(\lambda,\kappa))}$$

- And G is the solution of the linear integral equation

$$G(\lambda,\nu) = q^{-\alpha}\,\mathrm{cth}(\lambda-\nu-\eta) - \rho(\nu)\,\mathrm{cth}(\lambda-\nu) + \int_{\mathcal{C}} \mathrm{d}m(\mu)K_\alpha(\lambda-\mu)G(\mu,\nu)$$

The function ω – characterization by properties

Let $\zeta = \mathrm{e}^{\lambda}$, $\xi = \mathrm{e}^{\mu}$. Function $\zeta^{-\alpha}\omega(\lambda, \mu|\kappa, \alpha)$ is rational in ζ^2, i.e. a ratio of two polynomials $P(\zeta^2)/Q(\zeta^2)$. As a function of ζ^2 it is then characterized by e.g.:

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- Pole structure: $\zeta^{-\alpha}\omega(\lambda,\mu|\kappa,\alpha)$ has $N+2$ simple poles, N of which are located at the zeros of $\Lambda(\lambda|\kappa)$, the remaining two at $\zeta^2 = q^{\pm 2}\xi^2$
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- The degree of the polynomial in the numerator is $N+2$
- Normalization condition (JMS 09). Define

$$\Delta_\zeta f(\zeta) = f(q\zeta) - f(q^{-1}\zeta)\,, \quad \overline{D}_\zeta f(\zeta) = f(q\zeta) + f(q^{-1}\zeta) - 2\rho(\lambda) f(\zeta)$$

$$\widetilde{\omega}(\lambda,\mu|\kappa,\alpha) = \omega(\lambda,\mu|\kappa,\alpha) + \overline{D}_\zeta \overline{D}_\xi \Delta_\zeta^{-1}\psi(\zeta/\xi)$$

Then

$$\widetilde{\omega}(\beta_j,\mu|\kappa,\alpha) + \rho(\beta_j)\widetilde{\omega}(\beta_j - \eta,\mu|\kappa,\alpha) = 0\,, \quad j = 1,\dots,N$$

$$\lim_{\nu_j\to\infty} e^{\alpha(\nu_2-\nu_1)}\widetilde{\omega}(\nu_1,\nu_2|\kappa,\alpha) = 0\,, \quad j = 1,2$$

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$$\int_{\mathcal{C}}\mathrm{d}m(\mu)G(\mu,\nu_2)=\frac{q^{-\alpha}-\rho(\nu_2)}{q^{\alpha}-q^{-\alpha}}$$

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- Remark. Consider the following limits: 1. $\alpha\to 0$, 2. Trotter limit, 3. rational limit, 4. $\kappa\to 0$, 5. $T\to 0$. Then

$$\Psi_0^{XXX}(\nu_1,\nu_2)=\int_{-\infty}^{\infty}\mathrm{d}k\,\frac{\mathrm{e}^{-\mathrm{i}k(\nu_1-\nu_2)}}{1+\mathrm{e}^{|k|}}$$
$$=\mathrm{i}\partial_x \ln\left[\frac{\Gamma(\frac{1}{2}+\frac{\mathrm{i}x}{2})\Gamma(1-\frac{\mathrm{i}x}{2})}{\Gamma(\frac{1}{2}-\frac{\mathrm{i}x}{2})\Gamma(1+\frac{\mathrm{i}x}{2})}\right]_{x=(\nu_1-\nu_2)}$$

  is the derivative of the spinon-spinon scattering phase (Faddeev, Takhtadjan 81) satisfying the functional equation

$$\Psi_0^{XXX}(\nu_1,\nu_2)+\Psi_0^{XXX}(\nu_1-\mathrm{i},\nu_2)$$
$$=\frac{\mathrm{i}}{\nu_1-\nu_2}-\frac{\mathrm{i}}{\nu_1-\nu_2-\mathrm{i}}$$

  at the heart of the solution of the reduced q-Knizhnik-Zamolodchikov equation (BJMST 05)

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- The disorder parameter α modifies the linear integral equations. A factor of ρ appears in the measure which seems to complicate their analysis.
- Two examples
  1. Asymptotics of the function $G(\lambda, \nu)$ for $\lambda \to \infty$ (BG 10) and implications.
  2. Universal finite-size corrections for the isotropic Heisenberg chain (SABGKTT 11).

Asymptotics of G – the problem

- One of the functions we are interested in is the solution of the integral equation

$$G(\lambda,\nu) = q^{-\alpha} \operatorname{cth}(\lambda - \nu - \eta) - \rho(\nu) \operatorname{cth}(\lambda - \nu) + \int_{\mathcal{C}} \mathrm{d}m(\mu)\, K_\alpha(\lambda - \mu) G(\mu,\nu)$$

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- The limit $\lim_{\mathrm{Re}\,\lambda \to \infty} G(\lambda,\nu) = G(\infty,\nu)$ is not obvious from this equation. It merely implies the identity

$$(q^\alpha - q^{-\alpha}) \int_{\mathcal{C}} \mathrm{d}m(\lambda)\, G(\lambda,\nu) = q^{-\alpha} - \rho(\nu) - G(\infty,\nu)$$

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- In order to calculate this integral we introduced a function σ defined as the solution of the integral equation

$$\sigma(\lambda) = q^\alpha - q^{-\alpha} + \int_{\mathcal{C}} \mathrm{d}m(\mu)\, \sigma(\mu) K_\alpha(\mu-\lambda)$$

  because the dressed function trick applied to G and σ implies

$$(q^\alpha - q^{-\alpha}) \int_{\mathcal{C}} \mathrm{d}m(\lambda) G(\lambda,\nu) = \int_{\mathcal{C}} \mathrm{d}m(\lambda) \sigma(\lambda) \big(q^{-\alpha} \operatorname{cth}(\lambda-\nu-\eta) - \rho(\nu) \operatorname{cth}(\lambda-\nu)\big)$$

Asymptotics of G – the function ϕ

- In order to understand the properties of the 'dressed charge function' σ we decomposed the integral equation as

$$\sigma(\lambda) = q^{\alpha}\Big(1 + \int_{\mathcal{C}} \mathrm{d}m(\mu)\,\sigma(\mu)\,\mathrm{cth}(\lambda-\mu-\eta)\Big) - q^{-\alpha}\Big(1 + \int_{\mathcal{C}} \mathrm{d}m(\mu)\,\sigma(\mu)\,\mathrm{cth}(\lambda-\mu+\eta)\Big)$$

  This suggests the definition

$$\phi(\lambda+\eta) = 1 + \int_{\mathcal{C}} \mathrm{d}m(\mu)\,\sigma(\mu)\,\mathrm{cth}(\lambda-\mu+\eta)$$

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- Using the TQ-equation and the fact that all Bethe roots λ_k are located inside $\mathcal{C}$ we conclude that

$$\phi(\lambda+\eta) = 1 + \sum_{k=1}^{N/2} \frac{\sigma(\lambda_k)\,\mathrm{cth}(\lambda-\lambda_k+\eta)}{\rho(\lambda_k)\mathfrak{a}'(\lambda_k)}$$

  Thus,

$$\phi(\lambda) = 1 + \sum_{k=1}^{N/2} \frac{\sigma(\lambda_k)\,\mathrm{cth}(\lambda-\lambda_k)}{\rho(\lambda_k)\mathfrak{a}'(\lambda_k)}$$

  from which we can read off the analytic properties of $\phi(\lambda)$

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$$\mathrm{res}_{\lambda=\lambda_k}\,\phi(\lambda) = \frac{\sigma(\lambda_k)}{\rho(\lambda_k)\mathfrak{a}'(\lambda_k)}$$

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5. Using $\zeta = \mathrm{e}^{\lambda}$ we see that ϕ is the ratio of two polynomials of degree $N/2$ in ζ^2. Thus, $N/2+1$ complex numbers $\mu_1, \dots, \mu_{N/2}, C$ exist, such that

$$\phi(\lambda) = C \prod_{k=1}^{N/2} \frac{\mathrm{sh}(\lambda-\mu_k)}{\mathrm{sh}(\lambda-\lambda_k)}$$

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6. $$\sigma(\lambda) = q^{\alpha}\phi(\lambda-\eta) - q^{-\alpha}\phi(\lambda+\eta)$$

Asymptotics of G – the function r

- For arguments of ϕ inside $\mathcal{C}$ it is necessary to continue the integral analytically. Then,

$$\phi(\lambda) - \frac{q^{\alpha}\phi(\lambda-\eta) - q^{-\alpha}\phi(\lambda+\eta)}{\rho(\lambda)(1+\mathfrak{a}(\lambda))} = 1 + \int_{\mathcal{C}} \mathrm{d}m(\mu)\, \sigma(\mu)\, \mathrm{cth}(\lambda-\mu) = r(\lambda)$$

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- Two properties of the function $r(\lambda)$:

  (i) $r(\lambda)$ is regular inside $\mathcal{C}$

  (ii) $r(\lambda)$ has finite constant asymptotics for $\mathrm{Re}\,\lambda \to \pm\infty$

Asymptotics of G – the function r

- Using the definitions and of the functions ϕ and r we can now evaluate the integral

$$\int_{\mathcal{C}} \mathrm{d}m(\lambda)\,\sigma(\lambda)\big(q^{-\alpha}\operatorname{cth}(\lambda-\nu-\eta)-\rho(\nu)\operatorname{cth}(\lambda-\nu)\big) = \rho(\nu)r(\nu)-q^{-\alpha}\phi(\nu+\eta)+q^{-\alpha}-\rho(\nu)$$

It follows that

$$r(\nu)=\frac{q^{-\alpha}\phi(\nu+\eta)}{\rho(\nu)}-\frac{G(\infty,\nu)}{\rho(\nu)}$$

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It follows that

$$r(\nu) = \frac{q^{-\alpha}\phi(\nu+\eta)}{\rho(\nu)} - \frac{G(\infty,\nu)}{\rho(\nu)}$$

- This is roughly the way we went in our original derivation. At this point we used that we already knew that $G(\infty,\nu)=0$. Inserting this above we obtained

$$\Lambda(\lambda|\kappa+\alpha)Q(\lambda|\kappa)\phi(\lambda) \\ = q^{\kappa+\alpha}a(\lambda)Q(\lambda-\eta|\kappa)\phi(\lambda-\eta) + q^{-\kappa-\alpha}d(\lambda)Q(\lambda+\eta|\kappa)\phi(\lambda+\eta)$$

which is the TQ-equation for the function $Q(\lambda|\kappa)\phi(\lambda)$ with the known solution

$$Q(\lambda|\kappa)\phi(\lambda) = Q(\lambda|\kappa+\alpha)$$

From here we reversed the argument (BG 10)

Dressed charge and Q-function

- Summary: Setting

$$\phi(\lambda) = \frac{Q(\lambda|\kappa+\alpha)}{Q(\lambda|\kappa)}, \quad \sigma(\lambda) = q^{\alpha}\phi(\lambda-\eta) - q^{-\alpha}\phi(\lambda+\eta)$$

the ‘dressed charge function’ σ satisfies

$$\sigma(\lambda) = q^{\alpha} - q^{-\alpha} + \int_{\mathcal{C}} \mathrm{d}m(\mu)\,\sigma(\mu) K_{\alpha}(\mu-\lambda)$$

and can be also expressed as

$$\sigma(\lambda) = q^{\alpha} - q^{-\alpha} + \int_{\mathcal{C}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\,\phi(\mu) K_{\alpha}(\mu-\lambda)$$

- This allows us to calculate the asymptotics $\lim_{\mathrm{Re}\,\lambda\to\infty} G(\lambda,\nu) = 0$ and the asymptotics of $\widetilde{\omega}(\lambda,\mu|\kappa,\alpha)$ for $\mathrm{Re}\,\lambda \to \infty$.

A dual picture

- We can formulate a 'dual problem' for which everything but the last step is very similar as above. We define functions $\overline{G}$ and $\overline{\sigma}$ by

$$\overline{G}(\lambda,\nu) = -q^{\alpha}\,\mathrm{cth}(\lambda-\nu+\eta) + \rho(\lambda)\,\mathrm{cth}(\lambda-\nu) + \int_{\mathcal{C}} \mathrm{d}m(\mu)\,\overline{G}(\lambda,\mu)K_{\alpha}(\mu-\nu)\,,$$

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$$\overline{\sigma}(\lambda) = q^{-\alpha} - q^{\alpha} + \int_{\mathcal{C}} \mathrm{d}m(\mu)\,K_{\alpha}(\lambda-\mu)\overline{\sigma}(\mu)$$

- The asymptotics of $\overline{\sigma}$ is related with the asymptotics of σ,

$$\overline{\sigma}(\infty) = -\sigma(-\infty)\,, \quad \overline{\sigma}(-\infty) = -\sigma(\infty)$$

For $\overline{G}(\lambda,\nu)$ we know the asymptotics only for $\mathrm{Re}\,\lambda \to \infty$,

$$\overline{G}(\infty,\nu) = \frac{q^{\kappa}-q^{-\kappa}}{q^{\kappa}+q^{-\kappa}}\sigma(\nu)$$

but $\overline{G}(\nu,\infty)$ is unknown

A dual picture

As before we can introduce functions $\overline{\phi}$ and $\overline{r}$ with similar analytic properties and satisfying, in particular

1. $$\overline{\sigma}(\lambda) = q^{-\alpha}\overline{\phi}(\lambda - \eta) - q^{\alpha}\overline{\phi}(\lambda + \eta)$$

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But what is $\overline{r}(\lambda)$, what is $\overline{\phi}(\lambda)$, what is $\overline{G}(\lambda, \infty)$?

$\overline{\phi}$ for high temperature

- It follows from equation $(*)$ and from the similar equation for ϕ that

$$\overline{\phi}(\lambda)\big(q^{\alpha}\phi(\lambda-\eta)-q^{-\alpha}\phi(\lambda+\eta)\big)-\phi(\lambda)\big(q^{-\alpha}\overline{\phi}(\lambda-\eta)-q^{\alpha}\overline{\phi}(\lambda+\eta)\big) = \text{regular inside } \mathcal{C}$$

- This equation can be used to calculate the coefficients of the high-temperature expansion of of $\overline{\phi}$ in terms of those of the high-temperature expansion of ϕ

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- Same technique applies to the resolvent of the linear integral equations with our given kernel and measure. It can be expressed in terms of a function $\Phi(\lambda,\mu)$,

$$R(\nu_1,\nu_2) = K_\alpha(\nu_1-\nu_2) + \int_{\mathcal{C}} \frac{d\lambda}{2\pi i} \int_{\mathcal{C}} \frac{d\mu}{2\pi i} K_\alpha(\nu_1-\lambda)\Phi(\lambda,\mu)K_\alpha(\mu-\nu_2)$$

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$$R(\nu_1,\nu_2)=K_\alpha(\nu_1-\nu_2)+\int_{\mathcal{C}}\frac{d\lambda}{2\pi i}\int_{\mathcal{C}}\frac{d\mu}{2\pi i}K_\alpha(\nu_1-\lambda)\Phi(\lambda,\mu)K_\alpha(\mu-\nu_2)$$

- We wish to apply this technique in order to calculate the asymptotic expansion of $\widetilde{\omega}$ for large κ in the CFT-scaling limit

Finite-size corrections in the isotropic limit

THEOREM (SABGKTT 11). The finite size correction for all correlation functions of the isotropic Heisenberg chain behave like $1/L^2$.

For up to eight lattice sites we calculated them analytically (in terms of Riemann ζ-functions of integer arguments)

Finite-size corrections in the isotropic limit

Consider the finite length case (total length L) now (DGHK 07). Then in the isotropic limit with no external flux applied

$$\Psi^t(\xi_1,\xi_2) = 2\mathcal{K}(\xi_1-\xi_2) + \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\mathrm{ch}(\pi(\xi_2-x))}\left[\frac{g^{(+)}_{\xi_1}(x)}{1+\mathfrak{b}^{-1}(x)} + \frac{g^{(-)}_{\xi_1}(x)}{1+\overline{\mathfrak{b}}^{-1}(x)}\right]$$

where $g^{(\pm)}$ are the solutions of the integral equations

$$g^{(+)}_{\xi}(x) = \frac{-\pi}{\mathrm{ch}(\pi(\xi-x))} + \int_{-\infty}^{\infty} \frac{\mathrm{d}y\, g^{(+)}_{\xi}(y)}{2\pi(1+\mathfrak{b}^{-1}(y))}\mathcal{K}(x-y) - \int_{-\infty}^{\infty} \frac{\mathrm{d}y\, g^{(-)}_{\xi}(y)}{2\pi(1+\overline{\mathfrak{b}}^{-1}(y))}\mathcal{K}(x-y+\mathrm{i}-\mathrm{i}0)$$

$$g^{(-)}_{\xi}(x) = \frac{-\pi}{\mathrm{ch}(\pi(\xi-x))} + \int_{-\infty}^{\infty} \frac{\mathrm{d}y\, g^{(-)}_{\xi}(y)}{2\pi(1+\overline{\mathfrak{b}}^{-1}(y))}\mathcal{K}(x-y) - \int_{-\infty}^{\infty} \frac{\mathrm{d}y\, g^{(+)}_{\xi}(y)}{2\pi(1+\mathfrak{b}^{-1}(y))}\mathcal{K}(x-y-\mathrm{i}+\mathrm{i}0)$$

Finite-size corrections in the isotropic limit

the kernel is

$$\mathcal{K}(x) = \mathrm{i}\,\partial_x \ln \left[\frac{\Gamma\big(\frac{1}{2} + \frac{\mathrm{i}x}{2}\big)\Gamma\big(1 - \frac{\mathrm{i}x}{2}\big)}{\Gamma\big(\frac{1}{2} - \frac{\mathrm{i}x}{2}\big)\Gamma\big(1 + \frac{\mathrm{i}x}{2}\big)} \right]$$

and the auxiliary functions follow from

$$\ln \mathfrak{b}\ (x) = L \ln(\mathrm{th}(\pi x/2))$$

$$+ \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{2\pi} \mathcal{K}(x-y) \ln\big(1 + \mathfrak{b}\ (y)\big) - \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{2\pi} \mathcal{K}(x - y + \mathrm{i} - \mathrm{i}0) \ln\big(1 + \overline{\mathfrak{b}}\ (y)\big)$$

$$\ln \overline{\mathfrak{b}}\ (x) = L \ln(\mathrm{th}(\pi x/2))$$

$$+ \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{2\pi} \mathcal{K}(x-y) \ln\big(1 + \overline{\mathfrak{b}}\ (y)\big) - \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{2\pi} \mathcal{K}(x - y - \mathrm{i} + \mathrm{i}0) \ln\big(1 + \mathfrak{b}\ (y)\big)$$

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and the auxiliary functions follow from

$$\begin{aligned}\ln \mathfrak{b}_\delta(x) = & L\ln(\mathrm{th}(\pi x/2)) - \frac{\pi\delta}{\mathrm{ch}(\pi(\xi - x))} \\ & + \int_{-\infty}^{\infty}\frac{\mathrm{d}y}{2\pi}\mathcal{K}(x-y)\ln\bigl(1+\mathfrak{b}_\delta(y)\bigr) - \int_{-\infty}^{\infty}\frac{\mathrm{d}y}{2\pi}\mathcal{K}(x-y+\mathrm{i}-\mathrm{i}0)\ln\bigl(1+\overline{\mathfrak{b}}_\delta(y)\bigr)\end{aligned}$$

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For the large L analysis we modify these equations by introducing a new parameter δ

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Then

$$g_\xi^{(+)}(x) = \frac{\pi}{L} \frac{\partial_\delta \mathfrak{b}_\delta(x)}{\mathfrak{b}_\delta(x)}\bigg|_{\delta=0}, \quad g_\xi^{(-)}(x) = \frac{\pi}{L} \frac{\partial_\delta \overline{\mathfrak{b}}_\delta(x)}{\overline{\mathfrak{b}}_\delta(x)}\bigg|_{\delta=0}$$

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Defining

$$F(\xi_1, \xi_2) = \frac{2\delta L}{\pi} \mathcal{K}(\xi_1 - \xi_2) + \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\mathrm{ch}(\pi(\xi_2 - x))} \ln\left[(1 + \mathfrak{b}_\delta(x))(1 + \overline{\mathfrak{b}}_\delta(x))\right]$$

we obtain

$$\Psi^t(\xi_1, \xi_2) = \frac{\pi}{L} \partial_\delta F(\xi_1, \xi_2)\bigg|_{\delta=0}$$

Finite-size corrections in the isotropic limit

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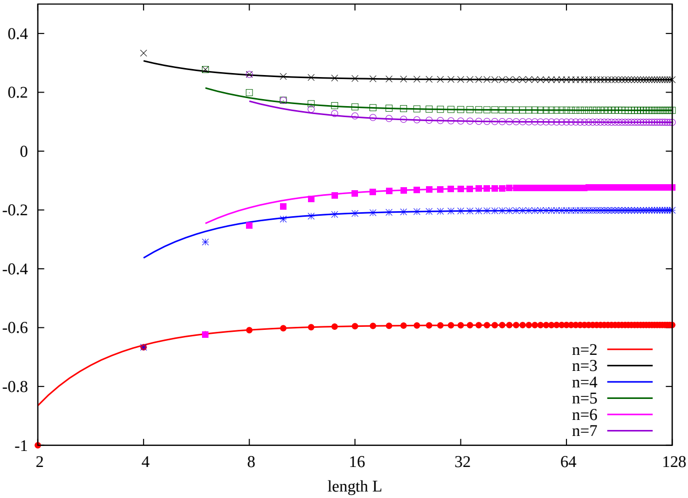
But F can be analyzed the same way as the dominant eigenvalue. Employing the dilog trick we obtain

$$\Psi^t(\xi_1, \xi_2) \sim 2\mathcal{K}(\xi_1 - \xi_2) - \frac{\pi^2}{3L^2} \mathrm{ch}(\pi(\xi_1 + \xi_2))$$

asymptotically for large L

Finite-size corrections in the isotropic limit



Length dependence of the two-point functions for $n = 2, \ldots, 7$

Asymptotics of asymptotics

The *zz*-correlation functions are of the form

$$\langle \sigma_1^z \sigma_{r+1}^z \rangle_L = \langle \sigma_1^z \sigma_{r+1}^z \rangle_\infty \left(1 + \frac{\gamma_r 4\pi^2}{L^2}\right)$$

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The two-point correlator for a primary field of scaling dimension x in CFT

$$C_r(L) = \frac{C}{r^{2x}} \left(\frac{\pi r/L}{\sin(\pi r/L)}\right)^{2x} \sim \frac{C}{r^{2x}} \left(1 + \frac{x\pi^2 r^2}{6L^2}\right)$$

For the *zz*-correlation functions we have to set $x = 1/2$, whence

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For the *zz*-correlation functions we have to set $x = 1/2$, whence

$$\gamma_r^{CFT} = \frac{r^2}{24}$$

r	1	2	3	4	5	6	7
γ_r	0.0470	0.1070	0.3268	0.5014	0.9013	1.1957	1.7761
γ_r/γ_r^{CFT}	1.1283	0.6419	0.8714	0.7521	0.8652	0.7971	0.8699

The coefficients γ_r for $r = 1, \dots, 7$ and the ratio of γ_r/γ_r^{CFT}

Conclusions

- The ‘physical part’ of the static correlation functions of the XXZ chain is determined by two functions, ρ and ω
- These can be described in terms of the solutions of well-behaved linear and non-linear integral equations
- The asymptotic analysis of their solutions may yield rather explicit results e.g. for short-range correlation functions
- Example: the static ground state correlation functions of the XXZ chain behave like $1/L^2$ for large L
- Yet, our understanding of these functions is still far from complete. We need to gain a better understanding of the integral equations when the disorder parameter α is non-zero. Then we might be able to obtain e.g. the large κ asymptotics in the CFT-scaling limit (cf. BJMS 10)