Bethe Ansatz solvable models out-of-equilibrium

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Outline

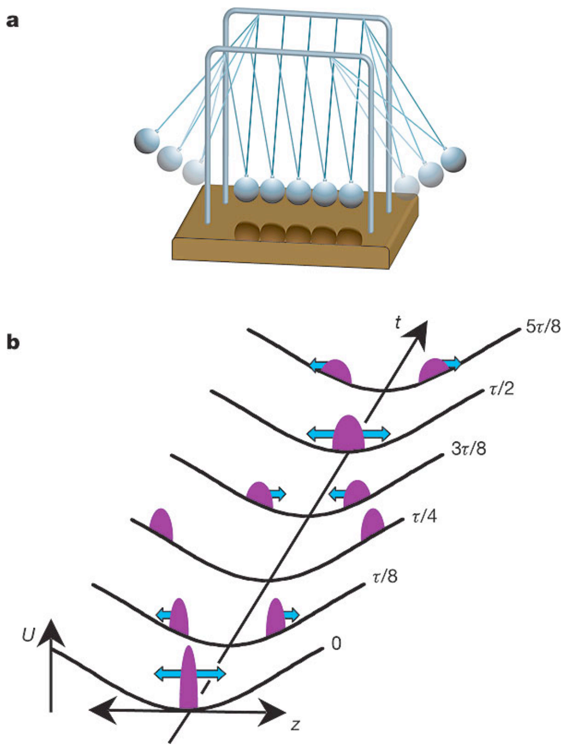
- Introduction
- Domain wall quench for the XXZ chain
- Interaction quench for the 1D Bose gas
- Conclusion and Outlook

Out-of-equilibrium dynamics

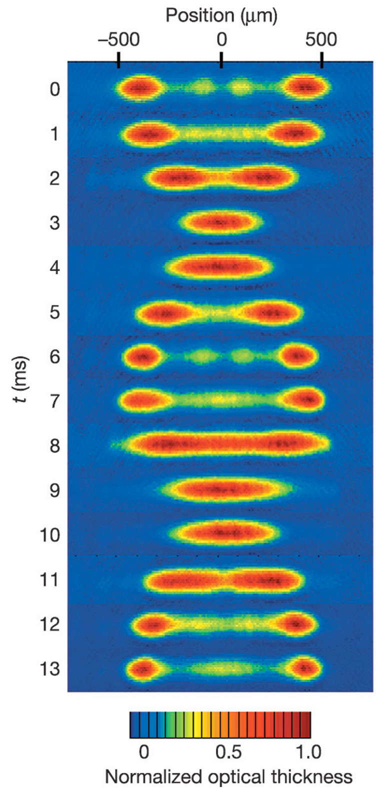
- Consider a many-body system with Hamiltonian *H*
- Prepare the system in a state $|\psi\rangle$ that is not an eigenstate
- Unitary time evolution $|\psi(t)\rangle = \exp(-iHt)|\psi\rangle$
- Study time evolution of observables $\langle\psi(t)|O(x)|\psi(t)\rangle$
- Coined a "quantum quench" by Calabrese and Cardy (PRL 2006)

Unitary time evolution in the lab

Quantum Newton's Cradle



Interacting Bose gas in a 1D optical trap



Kinoshita et al (Nature 2006)

Time evolution: brute force approach

$$|\psi(t)\rangle = \sum_n \langle n|\psi\rangle |n\rangle e^{-iE_n t} \qquad \sum_n |\langle n|\psi\rangle|^2 = 1$$

$$\langle\psi(t)|O(x)|\psi(t)\rangle = \sum_{n,m} \langle\psi|m\rangle\langle n|\psi\rangle\langle m|O(x)|n\rangle e^{i(E_m - E_n)t}$$

1. Construct eigenstates
2. Compute overlaps
3. Perform the sum over eigenstates
4. Compute matrix elements

Domain wall as initial state

JM and J-S Caux (NJP 2010)

Bosonization approach:

Lancaster and Mitra (PRE 2011)
Foster, Yuzbashyan and Altshuler (PRL 2010)

Domain wall: setup

- Initial state $|\phi\rangle = |\underbrace{\downarrow \ldots \downarrow}_{M} \underbrace{\uparrow \ldots \uparrow}_{N-M}\rangle$

- Unitary time evolution governed by

$$H_{XXZ} = J \sum_{j=1}^{N} \frac{1}{2\Delta} \left(S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^- \right) + S_j^z S_{j+1}^z$$

Overlap 1

- Recall $$S_j^- = \prod_{k=1}^{j-1} (A+D)(\xi_k) B(\xi_j) \prod_{l=j+1}^{N} (A+D)(\xi_l)$$

Kitanine et al (1999)

- Hence $$|\phi\rangle = |\downarrow \ldots \downarrow \uparrow \ldots \uparrow\rangle = \prod_{j=1}^{M} B(\xi_j)|0\rangle$$

- Note: general initial state is difficult

Overlap 2

- For an eigenstate $|\psi\rangle$ with rapidities $\{\lambda_j\}$

$$\langle\psi|\phi\rangle = \prod_{k=1}^{M}\prod_{l=1}^{M}\varphi(\lambda_l - \xi_k + i\eta)\frac{\det H(\{\lambda_j\},\{\xi_j\})}{\prod_{j>k}\varphi(\lambda_j-\lambda_k)\prod_{j<k}\varphi(\xi_j-\xi_k)}$$

$$H_{ab} = \frac{\varphi'(\lambda_a-\xi_b)}{\varphi(\lambda_a-\xi_b)} - \frac{\varphi'(\lambda_a-\xi_b+i\eta)}{\varphi(\lambda_a-\xi_b+i\eta)}$$

Special case of Slavnov's scalar product formula (1989)

Partition function 6-vertex model with domain wall boundary conditions, Izergin (1987)

Homogenous limit

- In the homogenous limit $\xi_j \to i\eta/2$

H is *M*-fold degenerate

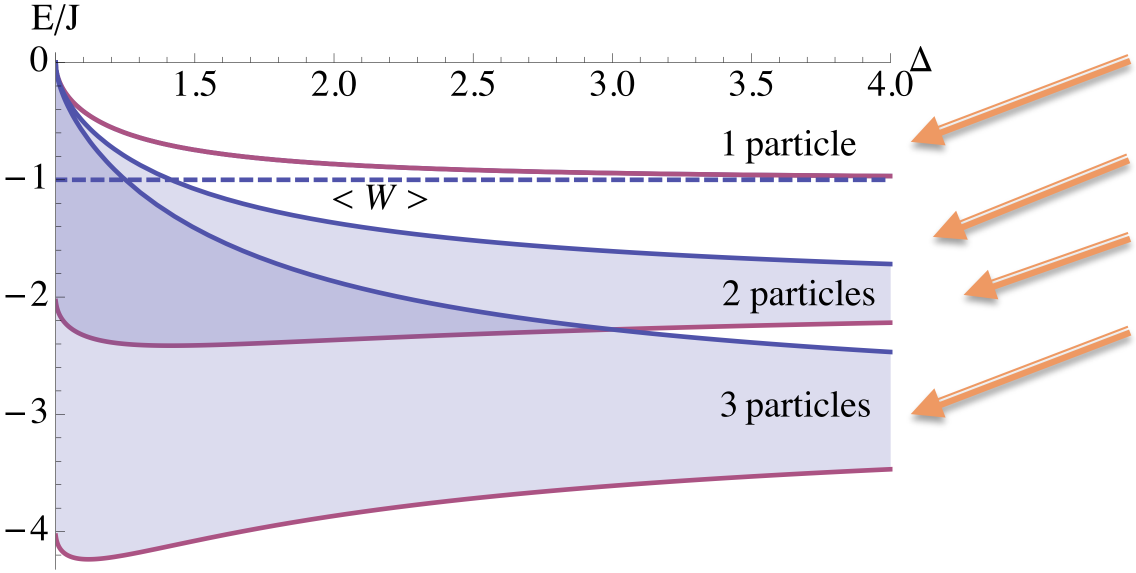
- Apply l'Hôpitals rule $H_{ab} \to \bar{H}_{ab} = \partial_{\xi_b}^{b-1} H_{ab}$

$$\partial_{\xi_b}^n H_{ab} = \sum_{j=0}^{n/2} c_j^n \left\{ \left(\frac{\varphi'(\lambda_a - \xi_b)}{\varphi(\lambda_a - \xi_b)} \right)^{2j+1} - \left(\frac{\varphi'(\lambda_a - \xi_b + i\eta)}{\varphi(\lambda_a - \xi_b + i\eta)} \right)^{2j+1} \right\}$$

- Keep only the highest power term with $c_{n/2}^n = n!$
- Results in $\langle \psi | \phi \rangle = \prod_{l=1}^{M} \varphi(\lambda_l + i\eta/2)^M \frac{\det \bar{H}}{\prod_{j>k} \varphi(\lambda_j - \lambda_k)}$

$$\bar{H}_{ab} = \left(\frac{\varphi'(\lambda_a - i\eta/2)}{\varphi(\lambda_a - i\eta/2)} \right)^b - \left(\frac{\varphi'(\lambda_a + i\eta/2)}{\varphi(\lambda_a + i\eta/2)} \right)^b$$

Spectral analysis



- M-strings
- 1,(M-1)-strings
- 2,(M-2)-strings
- 1,1,(M-2)-strings
- …

n-string $\lambda_j^n = \lambda^n + i\eta(n+1-2j)/2 \qquad j = 1 \ldots n$

$$\eta = \operatorname{acosh}(\Delta)$$

Overlap M-string

- Introduce $a_j \equiv \dfrac{\varphi'(\lambda_j + i\eta/2)}{\varphi(\lambda_j + i\eta/2)} \quad j = 1, \ldots, M+1$

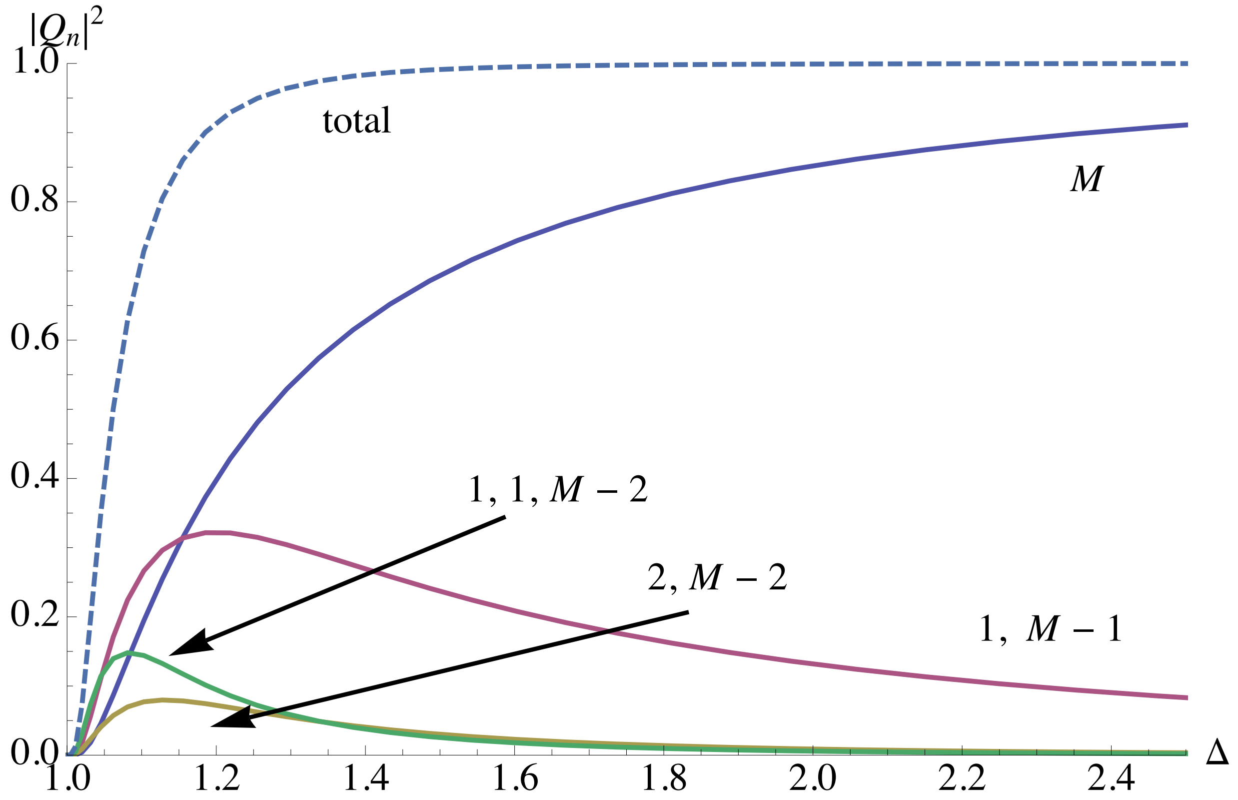
$$\det \bar{H} = \det \begin{pmatrix} a_2 - a_1 & \ldots & a_2^M - a_1^M \\ \vdots & \ddots & \vdots \\ a_{M+1} - a_M & \ldots & a_{M+1}^M - a_M^M \end{pmatrix} = \prod_{j<k} (a_j - a_k)$$

- Normalized overlap is

$$\frac{\langle \psi_M | \phi \rangle}{\sqrt{\langle \psi_M | \psi_M \rangle}} = \frac{\prod_{n=1}^{M-1} \varphi(in\eta)}{\varphi(\lambda_\alpha - i\eta M/2)^M} \sqrt{\frac{\varphi(\lambda_\alpha - i\eta M/2)\varphi(\lambda_\alpha + i\eta M/2)}{N}}$$

- (M-1,1)-string overlap is a sum of Vandermonde determinants, etc ….

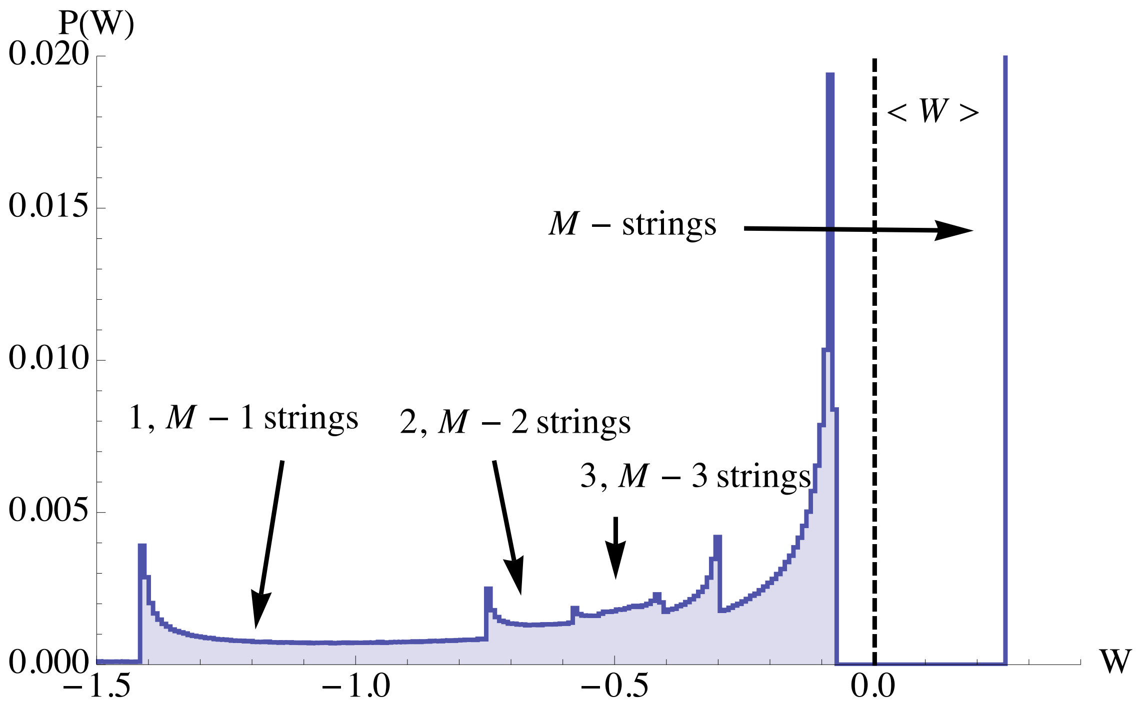
Overlap contributions



N=100, M=40

Work probability distribution

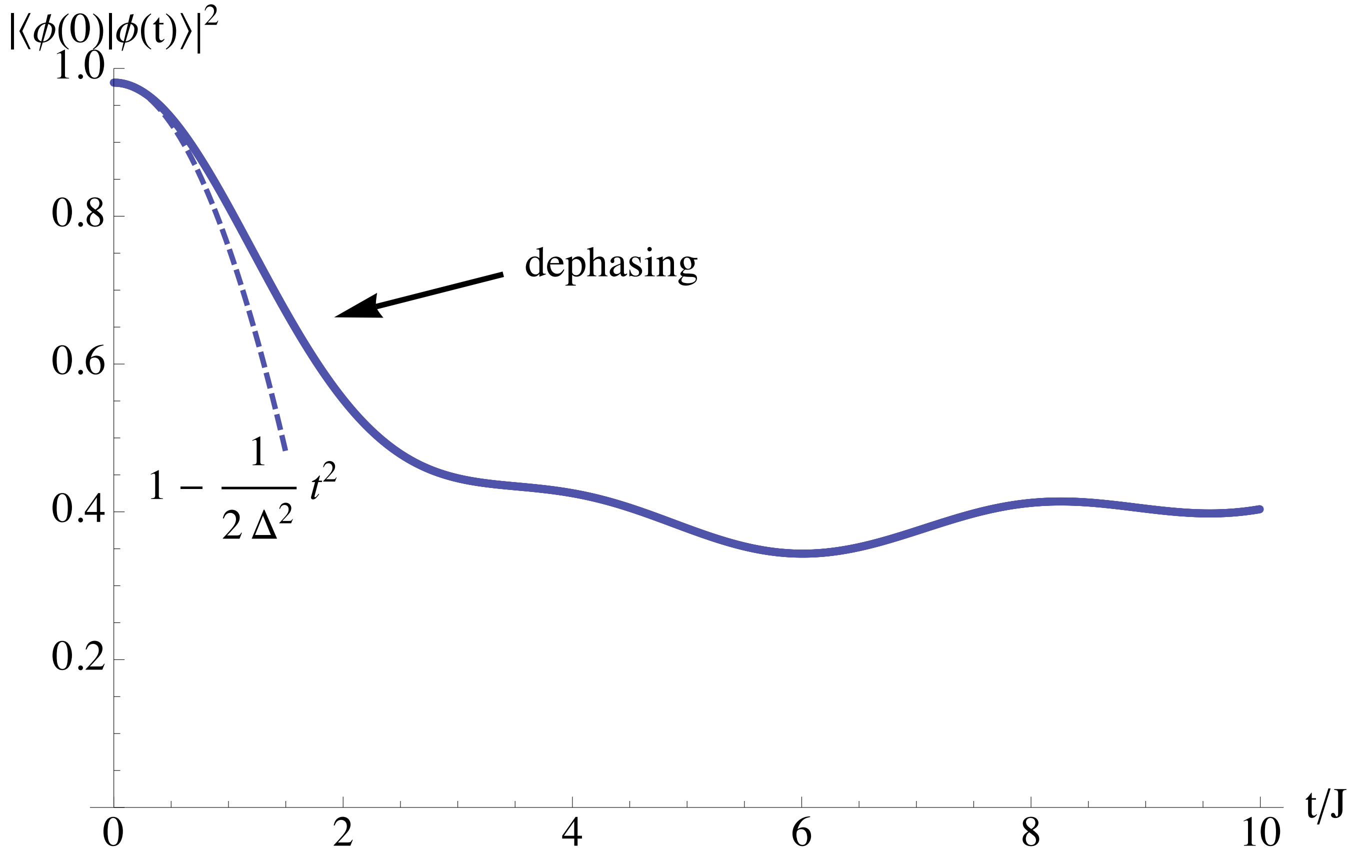
$$P(W) = \sum_n |\langle \phi | \Psi_n \rangle|^2 \delta(W - E_n + E_\phi)$$



Δ=1.5 N=250, M=100

Relaxation: Loschmidt echo

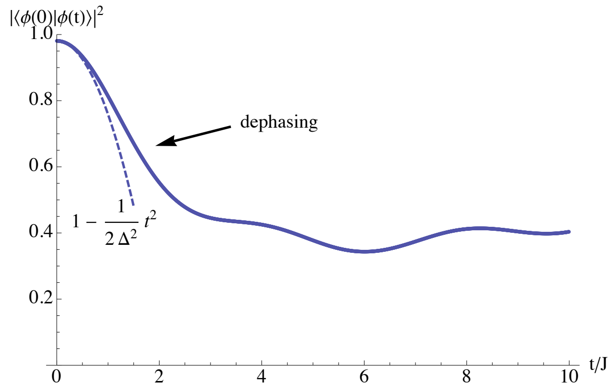
$$\mathcal{L}(t) = |\langle\phi(0)|\phi(t)\rangle|^2 = \sum_{m,n} e^{i(E_m - E_n)t} |\langle\phi|\psi_n\rangle|^2 |\langle\phi|\psi_m\rangle|^2$$

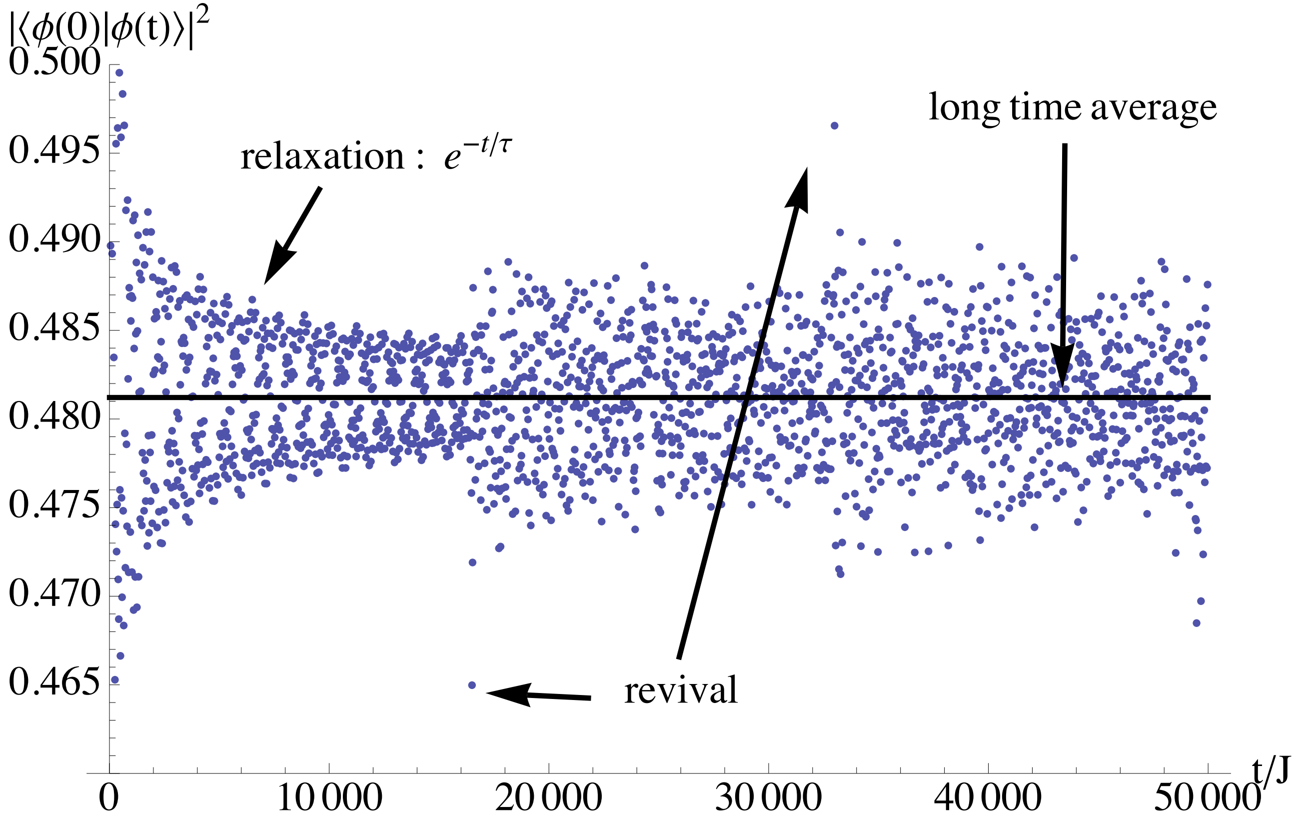


Δ=1.5 N=200, M=80

Relaxation: Loschmidt echo

$$\mathcal{L}(t) = \left|\langle\phi(0)|\phi(t)\rangle\right|^2 = \sum_{m,n} e^{i(E_m - E_n)t} |\langle\phi|\psi_n\rangle|^2 |\langle\phi|\psi_m\rangle|^2$$





Δ=1.5 N=200, M=80

Thermodynamic limit

- Summing all *M*-string overlaps

$$\lim_{N\to\infty}\sum_{\alpha=1}^{N}\frac{|\langle\psi_\alpha^M|\phi\rangle|^2}{\langle\psi_\alpha^M|\psi_\alpha^M\rangle}=\int_{-\pi/2}^{\pi/2}\frac{\prod_{n=1}^{M-1}|\varphi(in\eta)|^2}{N(\varphi(\lambda-i\eta M/2)\varphi(\lambda+iM\eta/2))^{M-1}}\rho_M(\lambda)d\lambda$$

$$\rho_M(\lambda)=\frac{N}{2\pi}\frac{d}{d\lambda^M}\theta_M(\lambda^M)$$

$$\lim_{M,N\to\infty}\sum_{\alpha=1}^{N}\frac{|\langle\psi_\alpha^M|\phi\rangle|^2}{\langle\psi_\alpha^M|\psi_\alpha^M\rangle}=\prod_{n=1}^{\infty}\left(1-e^{-2n\eta}\right)^2$$

Long time average

- Long time average of an observable:

$$\overline{\langle O \rangle} \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \phi(t)|O|\phi(t)\rangle dt$$

- Long time average of the Loschmidt echo:

$$\overline{\mathcal{L}} = \prod_{n=1}^{\infty} \left(1 - e^{-2n\eta}\right)^4, \qquad \Delta > 1$$

$$\eta = \mathrm{acosh}(\Delta)$$

Future work

- Define $|\psi^M\rangle = \sum_{\alpha=1}^{N} \langle\psi_\alpha^M|\phi\rangle\, |\psi_\alpha^M\rangle$

- In the limit the thermodynamic limit the state $|\psi^M\rangle$ coincides with the kink ground state of the ferromagnetic XXZ chain

- The density matrix of the kink ground state was obtained by Motegi and Sakai (PRE 2009)

- Can (M-1,1) excitations dealt with in a similar way?

Switching off the interactions in the 1D Bose gas

JM and JS Caux (unpublished)

Related work:

Imambekov et al (PR A2009)

1D Bose gas   Lieb & Liniger (1963)

$$H(c) = \int_0^L dx \left\{ \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c\, \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x) \right\}$$

Bosonic field operators $\quad [\Psi(x), \Psi^\dagger(y)] = \delta(x-y)$

- $c = 0$ free bosons
- $c = \infty$ Tonks-Girardeau limit 'free fermions'

Quench: $\quad c > 0 \rightarrow 0$

Time evolution: pair correlation

$$\langle \phi(t)|\Psi^\dagger(x)\Psi^\dagger(0)\Psi(x)\Psi(0)|\phi(t)\rangle$$

$$= \frac{1}{L^4}\sum_{k_1,k_2,k_3,k_4} e^{ix(k_1-k_3)}\langle\phi(t)|\Psi^\dagger_{k_1}\Psi^\dagger_{k_2}\Psi_{k_3}\Psi_{k_4}|\phi(t)\rangle$$

Time evolution of field operator

$$e^{iH(0)t}\Psi^\dagger_k e^{-iH(0)t} = \Psi^\dagger_k e^{ik^2 t}$$

since $H(0) = \sum_k k^2 \Psi^\dagger_k \Psi_k$

Time evolution: pair correlation

$$\langle \phi(t)|\Psi^\dagger(x)\Psi^\dagger(0)\Psi(x)\Psi(0)|\phi(t)\rangle$$

$$= \frac{1}{L^4}\sum_{k_1,k_2,k_3,k_4} e^{ix(k_1-k_3)}\langle \phi(t)|\Psi^\dagger_{k_1}\Psi^\dagger_{k_2}\Psi_{k_3}\Psi_{k_4}|\phi(t)\rangle$$

$$\langle \phi(t)|\Psi^\dagger_{k_1}\Psi^\dagger_{k_2}\Psi_{k_3}\Psi_{k_4}|\phi(t)\rangle =$$

$$e^{i(k_1^2+k_2^2-k_3^2-k_4^2)t}\ {}_c\langle GS|\Psi^\dagger_{k_1}\Psi^\dagger_{k_2}\Psi_{k_3}\Psi_{k_4}|GS\rangle_c$$

Form factor approach

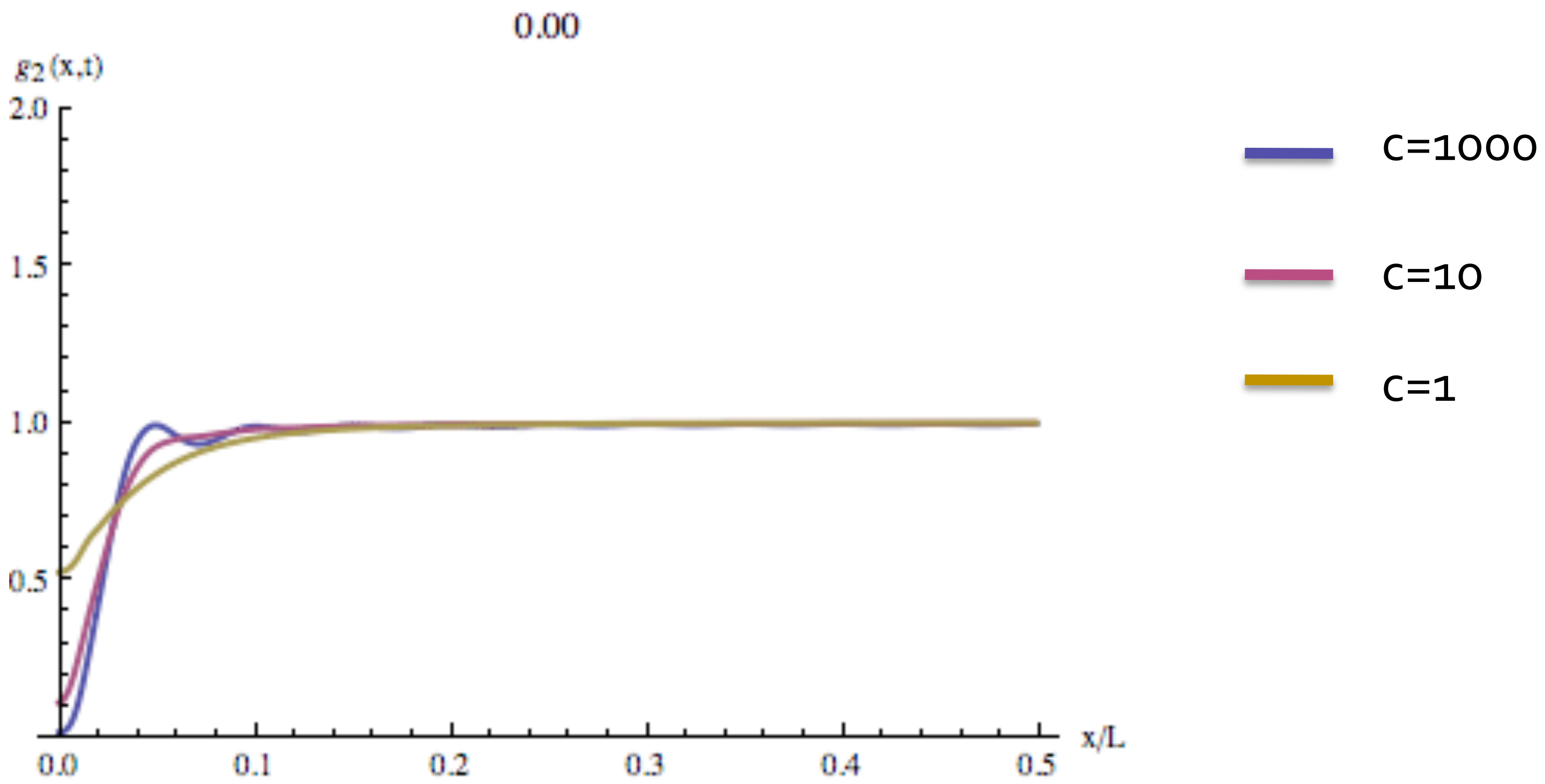
- Compute four-point function using form factors

$$\langle GS|\Psi^\dagger_{k_1}\Psi^\dagger_{k_2}\Psi_{k_3}\Psi_{k_4}|GS\rangle =$$
$$\sum_{n_1,n_2,n_3}\langle GS|\Psi^\dagger_{k_1}|n_1\rangle\langle n_1|\Psi^\dagger_{k_2}|n_2\rangle\langle n_2|\Psi_{k_3}|n_3\rangle\langle n_3|\Psi_{k_4}|GS\rangle$$

- Form factors for the two-point function were consider by Caux, Calabrese and Slavnov (JSTAT 2007)

Time evolution: pair correlation

Quench: $c \longrightarrow 0$ $(N, L = 20)$



Long time average and the canonical ensemble

- Integrate out fluctuations

$$\overline{\langle A \rangle} \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \phi(t)|A|\phi(t)\rangle dt$$

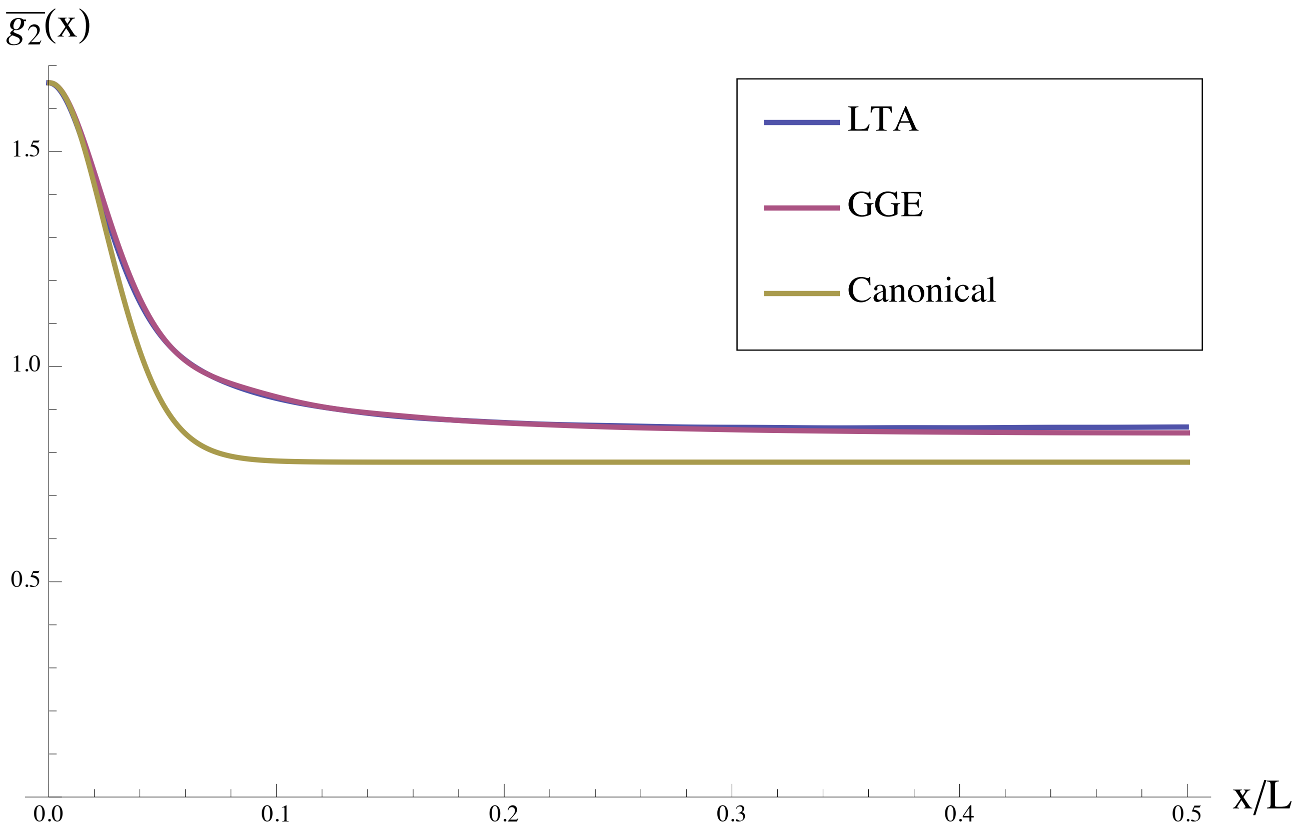
- Is it a thermal state?

$$\overline{\langle A \rangle} \stackrel{?}{=} \frac{\mathrm{Tr}\{Ae^{-\beta H_0}\}}{\mathrm{Tr}\{e^{-\beta H_0}\}}$$

- Determine β via $\langle \phi(0)|H_0|\phi(0)\rangle = \dfrac{\mathrm{Tr}\left\{H_0 e^{-\beta H_0}\right\}}{\mathrm{Tr}\left\{e^{-\beta H_0}\right\}}$

Long time average and the canonical ensemble

- Only local thermal behavior



Long time average and the Generalized Gibbs Ensemble (Rigol et al, 2007)

- Use a set of 'relevant' conserved charges: $\Psi_k^\dagger \Psi_k$

$$\overline{\langle A \rangle} \stackrel{?}{=} \frac{\mathrm{Tr}\left\{A\, e^{-\sum_k \beta_k\; k^2 \Psi_k^\dagger \Psi_k}\right\}}{\mathrm{Tr}\left\{e^{-\sum_k \beta_k\; k^2 \Psi_k^\dagger \Psi_k}\right\}}$$

Determine β_k from the initial conditions

- Every momentum mode acquires its own inverse temperature β_k

Open problems

- More overlap formulas, for instance

$$\langle \psi(\Delta_2) | \phi(\Delta_1) \rangle$$

$$\langle \psi(\Delta) | S^{-}_{j_1} \dots S^{-}_{j_M} | 0 \rangle$$

- Are there quench problems where the summation over eigenstates can be done efficiently?