Form factor approach to the correlation functions of critical models

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- "Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions" *J. Stat. Mech. P04003 (2009)*
- "On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain" *J. Math. Phys. 50, 095209 (2009)*
- "On the thermodynamic limit of particle-holes form factors in the massless XXZ Heisenberg chain" *arXiv:1003.4557*
- "Form factor approach to the correlation functions of critical models" *to appear*

Correlation functions of critical (integrable) models

- **Asymptotic results** $\langle \sigma_1^{\alpha} \sigma_m^{\beta} \rangle \underset{m \to \infty}{\sim}$?
  - Luttinger liquid approximation / C.F.T. and finite size effects
    Luther and Peschel, Haldane, Cardy, Affleck, ... Lukyanov, ...
- **Exact results (XXZ, ...)**
  - Free fermion point $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa ...
  - From 1984: Izergin, Korepin ... (first attempts using Bethe ansatz for general Δ)
  - General Δ: multiple integral representations (for building blocks)
    ⋆ 1992-96 Jimbo, Miwa ... → from q-vertex op. and qKZ eq.
    ⋆ 1999 Kitanine, Maillet, Terras → from Algebraic Bethe Ansatz
  - Several developments for the last ten years: Kitanine, Maillet, Slavnov, Terras; Boos, Korepin, Smirnov; Boos, Jimbo, Miwa, Smirnov, Takeyama; Göhmann, Klümper, Seel; Caux, Hagemans, Maillet ...

↪ **Asymptotic behavior from exact results ?**

Our favorite example : the XXZ Heisenberg chain

The *XXZ* spin-1/2 Heisenberg chain **in a magnetic field** is a quantum interacting model defined on a one-dimensional lattice with M sites, with Hamiltonian,

$$H_{XXZ} = \sum_{m=1}^{M} \{\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1)\} - \mathbf{h} \sum_{m=1}^{M} \sigma_m^z$$

Quantum space of states : $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$, $\dim \mathcal{H} = 2^M$.

$\sigma_m^{x,y,z}$: local spin operators (in the spin-$\frac{1}{2}$ representation) at site m
They act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.

- periodic boundary conditions
- disordered regime, $|\Delta| < 1$ and $h < h_c$

Correlation function : general strategies

At zero temperature only the ground state $|\omega\rangle$ contributes :

$$g_{12} = \langle\omega|\theta_1\theta_2|\omega\rangle$$

Two main strategies to evaluate such a function:

(i) compute the action of local operators on the ground state $\theta_1\theta_2|\omega\rangle = |\tilde{\omega}\rangle$ and then calculate the resulting scalar product:

$$g_{12} = \langle\omega|\tilde{\omega}\rangle$$

(ii) insert a sum over a complete set of eigenstates $|\omega_i\rangle$ to obtain a sum over one-point matrix elements (form factor type expansion) :

$$g_{12} = \sum_i \langle\omega|\theta_1|\omega_i\rangle \cdot \langle\omega_i|\theta_2|\omega\rangle$$

Algebraic Bethe ansatz and correlation functions

1. **Diagonalise the Hamiltonian using ABA**
   $\rightarrow$ key point : Yang-Baxter algebra $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$
   $\rightarrow$ $|\,\psi_g\,\rangle = B(\lambda_1)\dots B(\lambda_N)|\,0\,\rangle$ with $\mathcal{Y}(\lambda_j; \{\lambda\}) = 0$ (Bethe eq.)
2. **Act with local operators on eigenstates**
   $\rightarrow$ solve the quantum inverse problem (1999):
   $$\sigma_j^{\alpha_j} = f_j^{\alpha_j}(A, B, C, D) = \prod(A, B, C, D)$$
   $\rightarrow$ use Yang-Baxter commutation relations
3. **Compute the resulting scalar products** (determinant representation)
   $\rightarrow$ determinant representation for form factors of the finite chain
   $\rightarrow$ elementary building blocks of correlation functions as multiple integrals in the thermodynamic limit (2000)
4. **Two-point function:** sum up elementary blocks or form factors?
   $\rightarrow$ **Master equation representation** for the finite chain (2005)
5. **Asymptotic analysis of the two-point function (2008-2010):**
   $\rightarrow$ Series expansion of the Master equation and asymptotic analysis

Results for XXZ from master equation

Generating function

$$Q^{\kappa}_{1,m} = \prod_{n=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma^z_n \right)$$

Asymptotic behavior

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \underbrace{G^{(0)}(\beta, m)[1+o(1)]}_{\text{non-oscillating terms}} + \underbrace{\sum_{\sigma=\pm} G^{(0)}(\beta + 2i\pi\sigma, m)[1+o(1)]}_{\text{oscillating terms}}$$

$$G^{(0)}(\beta, m) = C(\beta)\, e^{m\beta D}\, m^{\frac{\beta^2}{2\pi^2} Z(q)^2}$$

- $Z(\lambda)$ is the dressed charge $\quad Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda - \mu)\, Z(\mu) = 1$
- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu)\mathrm{d}\mu = \frac{1 - \langle \sigma^z \rangle}{2} = \frac{p_F}{\pi}$
- The coefficient $C(\beta)$ is given as the ratio of four Fredholm determinants.
- sub-leading oscillating terms restore the $2\pi i$-periodicity in β

2-point function asymptotic behavior

$$\langle \sigma^z_1 \sigma^z_{m+1} \rangle = (2D-1)^2 - \frac{2Z(q)^2}{\pi^2 m^2} + 2|F_{\sigma^z}|^2 \cdot \frac{\cos(2mp_F)}{m^{2Z(q)^2}} + o\left(\frac{1}{m^2}, \frac{1}{m^{2Z(q)^2}} \right)$$

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Form factors strike back!

The umklapp form factor

$$\lim_{N,M\to\infty}\left(\frac{M}{2\pi}\right)^{2\mathcal{Z}^2}\frac{|\langle\psi(\{\mu\})|\sigma^z|\psi(\{\lambda\})\rangle|^2}{\|\psi(\{\mu\})\|^2\cdot\|\psi(\{\lambda\})\|^2}=|F_{\sigma^z}|^2.$$

with

$$2\mathcal{Z}^2=Z(q)^2+Z(-q)^2$$

- $\{\lambda\}$ are the Bethe parameters of the ground state
- $\{\mu\}$ are the Bethe parameters for the excited state with one particle and one hole on opposite sides of the Fermi boundary (umklapp type excitation).

$\hookrightarrow$ Higher terms in the asymptotic expansion will involve n - particle/holes form factors corresponding to $2np_F$ oscillations
$\hookrightarrow$ Properly normalized form factors will be related to the corresponding amplitudes

$\hookrightarrow$ It strongly suggests that there should be a way to analyze the asymptotic behavior of the correlation function directly from the form factor series!

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The form factor series for critical models

↪Our goal is to study the behavior of correlation functions in critical models using their form factor expansion

$$\langle \mathcal{O}_1(x')\mathcal{O}_2(x'+x)\rangle = \sum_{|\,\psi'\rangle} \mathcal{F}^{(1)}_{\psi_g\,\psi'}(x')\,\mathcal{F}^{(2)}_{\psi'\,\psi_g}(x+x')$$

↪ Main difficulty : form factors scale to zero in the infinite size limit ($L \to \infty$) for critical models as can be seen in various examples like XXZ or Bose gas; in fact this simply reflect the corresponding conformal dimensions of the local operators involved and of the creation operators for the excited state ψ'.

$$\mathcal{F}^{(1)}_{\psi_g\,\psi'}(x')\cdot\mathcal{F}^{(2)}_{\psi'\,\psi_g}(x+x') = L^{-\theta}\, e^{ix\mathcal{P}_{ex}}\,\mathcal{A}(\psi'\,,\psi_g)$$

↪ Analyze the form factor series for large (but finite) system size, for example in the asymptotic regime where the distance x between local operators becomes large; the thermodynamic limit will be taken only at the end of the computation.

↪ Hence we need to describe states that will contribute to the leading behavior of the series in the limits $x \to \infty$ and $L \to \infty$ with $x << L$, and also to compute the corresponding form factors and their behavior in these limits

The particle-hole spectrum

$\hookrightarrow$ We study first the correlation functions of quantum integrable models that are solvable by the Bethe Ansatz in their critical regime and such that the ground state solution of the Bethe equations can be described in terms of real rapidities λ_j densely filling (with a density $\rho(\lambda)$) the Fermi zone $[-q, q]$:

$$Lp_0(\lambda_j) - \sum_{k=1}^{N} \vartheta(\lambda_j - \lambda_k) = 2\pi \left(j - \frac{N+1}{2} \right), \qquad j = 1, \dots, N.$$

$\hookrightarrow$ and excited states parametrized by numbers $\{\mu_{\ell_a}\}_1^{N'}$ with $N' = N + k$, k finite and fixed, involving other choices of integers $\ell_1 < \cdots < \ell_{N'}$ in the *rhs* :

$$Lp_0(\mu_{\ell_j}) - \sum_{k=1}^{N'} \vartheta(\mu_{\ell_j} - \mu_{\ell_k}) = 2\pi \left(\ell_j - \frac{N'+1}{2} \right), \qquad j = 1, \dots, N'.$$

$$\ell_a = a \ , \ a \in \{1, \dots, N'\} \setminus \{h_1, \dots, h_n\} \qquad \ell_{h_a} = p_a \ , \ p_a \in \mathbb{Z} \setminus \{1, \dots, N'\}.$$

$\hookrightarrow$To each such a choice of integers $\{p_a\}$ and $\{h_a\}$ there is associated a configuration of rapidities for the particles $\{\mu_{p_a}\}$ and for the holes $\{\mu_{h_a}\}$

The shift function for excited states

↪The shift function describes the spacing between the root (pseudo-momentum) λ_j for the ground state in the N sector and the corresponding parameters μ_j (for the same integer j) for the excited state:

$$\mu_j - \lambda_j = \frac{F(\lambda_j)}{\rho(\lambda_j) L} + O(L^{-2}),$$

↪ For integrable models this shift function characterize the various excited states and can be shown to satisfy in the thermodynamic limit linear integral equation involving the Lieb Kernel and the bare scattering phase.

↪ For non-integrable models this shift function still exists and gives the way the particles remaining in the Fermi zone get shifted (due to interactions) from their positions in the ground state. Its value at the Fermi boundary in the thermodynamic limit will be shown to characterize the various critical exponents.

The form factor structure for critical models

The form factors in the thermodynamic limit

$$\mathcal{F}^{(1)}_{\psi_g\,\psi'}(x') \cdot \mathcal{F}^{(2)}_{\psi'\,\psi_g}(x+x') = L^{-\theta}\, e^{ix\mathcal{P}_{ex}}\, \mathcal{S}\, \mathcal{D}.$$

The discrete part $\mathcal{D}$ has a purely kinematical interpretation and its functional form is universal and model independent (in fact it holds also for a large class of non-integrable models). It depends on the rapidities $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$ *and* on the integers $\{p_a\}$ and $\{h_a\}$.

The smooth part $\mathcal{S}$ and the exponent θ are model dependent. The smooth part depends on the rapidities $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$.

$\hookrightarrow$ It is *solely* the kinematical factor $\mathcal{D}$ (together with the values of θ for the various form factors) that drives the long-distance asymptotic behavior, while the $\mathcal{S}$ part enters only the corresponding amplitude.

The form factor series in the large distance limit (1)

$$\langle \mathcal{O}_1(x')\,\mathcal{O}_2(x+x')\rangle_{ph} = \lim_{L\to\infty} \sum_{\{\mu_p\},\{\mu_h\}} L^{-\theta}\, e^{ix\mathcal{P}_{ex}}\, \mathcal{S}(\{\mu_p\},\{\mu_h\})\,\mathcal{D}(\{\mu_p\},\{\mu_h\}|\{p\},\{h\}).$$

In the large distance limit $x \to \infty$, the oscillatory character of these sums localizes them, in the absence of any other saddle point of the oscillating exponent (such saddle points will appear in the time dependent case), around small excitations on the Fermi boundaries $\pm q$.

$\hookrightarrow$Hence we consider so-called critical n particle-hole excited state $\{\mu_{\ell_a}\}$ for which the rapidities $\{\mu_{p_a}\}$ and $\{\mu_{h_a}\}$ of such a state accumulate on the two end-points of the Fermi-zone in the thermodynamic limit. Form factors corresponding to any such a state are called critical form factors

In the thermodynamic limit, there will be $n_p^{\pm}$ particles whose rapidities are equal to $\pm q$ and $n_h^{\pm}$ holes whose rapidities are equal to $\pm q$ with $n_p^+ + n_p^- = n_h^+ + n_h^- = n$. A given critical excited state belong to the $\mathbf{P}_\ell$ class if the distribution of particles and holes on the Fermi boundaries is such that

$$n_p^+ - n_h^+ = n_h^- - n_p^- = \ell, \qquad -n \le \ell \le n.$$

Then such an excited state has momentum $2\ell p_F$ in the thermodynamic limit and its associated critical form factor will also be said to belong to the $\mathbf{P}_\ell$ class.

The form factor series in the large distance limit (2)

Localization of the form factor series by analogy with the multiple integral situation :

$$I_n(x) = \int\limits_{\mathbb{R}\setminus[-q,q]} d^n\mu_p \int\limits_{[-q,q]} d^n\mu_h \, f(\{\mu_p\},\{\mu_h\}) \prod_{j=1}^{n} e^{ix(p(\mu_{p_j})-p(\mu_{h_j}))}, \qquad x \to \infty,$$

$\hookrightarrow$ If $f(\{\mu_p\},\{\mu_h\})$ is a holomorphic function and $p(\mu)$ has no saddle points on the integration contours then the large x asymptotic analysis reduces to the calculation of the integral in small vicinities of the endpoints, where $f(\{\mu_p\},\{\mu_h\})$ can be replaced by $f(\{\pm q\},\{\pm q\})$.

$\hookrightarrow$ If $f(\{\mu_p\},\{\mu_h\})$ has integrable singularities at $\pm q$, for example $f(\{\mu_p\},\{\mu_h\}) = (q-\mu_{h_1})^{\nu_+}(\mu_{h_1}+q)^{\nu_-} f_{reg}(\{\mu_p\},\{\mu_h\})$, then one has to keep the singular factors $(q \mp \mu_{h_1})^{\nu_\pm}$ as they are but we can replace the regular part $f_{reg}(\{\mu_p\},\{\mu_h\})$ by an appropriate constant $f_{reg}(\{\pm q\},\{\pm q\})$.

The form factor series in the large distance limit (3)

$\hookrightarrow$ By analogy with these oscillating multiple integrals we obtain:

- the smooth part of the form factor $\mathcal{S}(\{\mu_p\},\{\mu_h\})$ doesn't introduce any singularities and thus, the corresponding rapidities can be set equal to their values in the $\mathbf{P}_\ell$ class. Likewise can be done for the part of the discrete form factor depending smoothly on the rapidities $\mathcal{D}(\{\mu_p\},\{\mu_h\}|\{p\},\{h\})$.
- we need to keep and to treat explicitly the summation of the integer dependent part of $\mathcal{D}(\{\mu_p\},\{\mu_h\}|\{p\},\{h\})$.

$\hookrightarrow$ Thus, the main contribution to the asymptotic behavior of the correlation function is produced by the critical form factors. Hereby the smooth part becomes a constant depending only on the $\mathbf{P}_\ell$ class of the form factor. The discrete part $\mathcal{D}$ plays the role of the singular factors $(q \mp \mu_{h_1})^{\nu_\pm}$ in the integral. Hence fixing the $\mathbf{P}_\ell$ class of critical from factors we should still take the sum over all the excited states within this class.

Warning: The form factor sums cannot have the sense of integral sums even for L large. Indeed, it produces eventually the factor L^{θ_ℓ} that makes the final result finite; however, θ_ℓ being in general not an integer, such a sum hardly reduces to a Riemann sum. This feature explains the difficulties already noted in the literature while trying to use the form factor approach directly in the continuum limit for massless models.

Momentum of critical states

Critical states are gathered into various classes $\mathbf{P}_\ell$ having momentum $2\ell p_F$, where $p_F = \pi D$ is the Fermi momentum. Different states belonging to a given $\mathbf{P}_\ell$ class can be characterized by sets of integers $\{p_a^\pm\}_1^n$ and $\{h_a^\pm\}_1^{n'}$ with $n - n' = \ell$. We introduce the reparametrization :

$$\begin{array}{ll} p_j = p_j^+ + N, & \text{if} \quad \mu_{p_j} = q\,, \\ p_j = 1 - p_j^-\,, & \text{if} \quad \mu_{p_j} = -q\,, \\ h_j = N + 1 - h_j^+, & \text{if} \quad \mu_{h_j} = q\,, \\ h_j = h_j^-\,, & \text{if} \quad \mu_{h_j} = -q\,. \end{array}$$

$$\mathcal{P}_{ex} = 2\ell p_F + \frac{2\pi}{L}\mathcal{P}_{ex}^{(d)}.$$

$$\mathcal{P}_{ex}^{(d)} = \sum_{j=1}^{n_p^+} p_j^+ - \sum_{j=1}^{n_p^-} p_j^- + \sum_{j=1}^{n_h^+} h_j^+ - \sum_{j=1}^{n_h^-} h_j^- + n_p^- - n_h^+\,.$$

Critical exponents

It can be shown that the power θ and the smooth part of all form factors belonging to a given class $\mathbf{P}_\ell$ is constant and is given by the value of the shift function at the Fermi boundaries. This is a very general result that can be proved whenever one has a Cauchy determinant part in the representations for the form factors. In the case of a generic particle/hole excited state, the shift function depends on the rapidities of the particles $\{\mu_p\}$ and holes $\{\mu_h\}$ ocuring in the given state. However, in the case of critical excited states, this shift function $F_\ell(\lambda)$ solely depends on the index ℓ. It is thus the same for all the critical excited states belonging to the $\mathbf{P}_\ell$ class.

$$F_{\ell,-} \equiv F_\ell(-q), \quad F_{\ell,+} \equiv F_\ell(q) + N' - N.$$

In particular the exponent θ for the class $\mathbf{P}_\ell$ can be written as

$$\theta_\ell = (F_{\ell,-} + \ell)^2 + (F_{\ell,+} + \ell)^2.$$

The critical form factor series (1)

$$\langle \mathcal{O}_1(x')\,\mathcal{O}_2(x+x')\rangle_{cr} = \lim_{L\to\infty} \sum_{\ell=-\infty}^{\infty} L^{-\theta_\ell}\, e^{2ix\ell p_F}\, \mathcal{S}_\ell \sum_{\substack{\{p\},\{h\}\\ n_p^+ - n_h^+ = \ell}} e^{\frac{2\pi i x}{L}\mathcal{P}_{ex}^{(d)}} \mathcal{D}^{(\ell)}(\{p\},\{h\}).$$

Using the fact that the smooth part is the same for all the critical form factors of the $\mathbf{P}_\ell$ class, it is easy to notice that it can be expressed in terms of the simplest form factor of the class: corresponding to the ℓ-shifted $|\,\psi'_\ell\,\rangle$ with the Bethe roots given by the following equations:

$$Lp_0(\mu_j) - \sum_{k=1}^{N'} \vartheta(\mu_j - \mu_k) = 2\pi\left(j + \ell - \frac{N'+1}{2}\right), \qquad j = 1, \ldots, N'.$$

Let's consider the product of form factors $\overline{\mathcal{F}}_\ell^{(1)}\mathcal{F}_\ell^{(2)}$ which correspond to matrix elements of the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ between the ground state and the ℓ-shifted state $|\,\psi'_\ell\,\rangle$:

$$\overline{\mathcal{F}}_\ell^{(1)}\mathcal{F}_\ell^{(2)} = \lim_{L\to\infty} L^{\theta_\ell}\, \frac{\langle\,\psi_g\,|\mathcal{O}_1(x')|\,\psi'_\ell\,\rangle\langle\,\psi'_\ell\,|\mathcal{O}_2(x')|\,\psi_g\,\rangle}{\langle\,\psi_g\,|\psi_g\rangle\langle\,\psi'_\ell\,|\psi'_\ell\rangle}$$

The reduced critical form factors

$$\mathcal{F}^{(1)}_{\psi_g\psi'}\mathcal{F}^{(2)}_{\psi_g\psi'} = \frac{\overline{\mathcal{F}}^{(1)}_{\ell}\mathcal{F}^{(2)}_{\ell}}{L^{\theta_\ell}} \frac{G^2(1+F_{\ell,+})G^2(1-F_{\ell,-})}{G^2(1+\ell+F_{\ell,+})G^2(1-\ell-F_{\ell,-})}$$

$$\left(\frac{\sin(\pi F_{\ell,+})}{\pi}\right)^{2n_h^+}\left(\frac{\sin(\pi F_{\ell,-})}{\pi}\right)^{2n_h^-}$$

$$\times R_{n_p^+,n_h^+}(\{p^+\},\{h^+\}|F_{\ell,+})\, R_{n_p^-,n_h^-}(\{p^-\},\{h^-\}|-F_{\ell,-}),$$

where $G(z)$ is the Barnes function satisfying $G(z+1)=\Gamma(z)G(z)$, and

$$R_{n,n'}(\{p\},\{h\}|F) = \frac{\prod_{j>k}^{n}(p_j-p_k)^2\prod_{j>k}^{n'}(h_j-h_k)^2}{\prod_{j=1}^{n}\prod_{k=1}^{n'}(p_j+h_k-1)^2}\,\Gamma^2\left(\begin{array}{c} \{p_k+F\}\ ,\ \{h_k-F\} \\ \{p_k\}\ ,\ \{h_k\} \end{array}\right).$$

with

$$\Gamma\left(\begin{array}{c} a_1\ ,\ \ldots\ ,\ a_r \\ b_1\ ,\ \ldots\ ,\ b_s \end{array}\right) = \prod_{k=1}^{r}\Gamma(a_k)\cdot\prod_{k=1}^{s}\Gamma(b_k)^{-1}.$$

The critical form factor series (2)

$$\langle \mathcal{O}_1(x')\,\mathcal{O}_2(x+x')\rangle_{cr} = \lim_{L\to\infty} \sum_{\ell=-\infty}^{\infty} L^{-\theta_\ell}\, e^{2ix\ell k_F}\, \overline{\mathcal{F}}_\ell^{(1)} \mathcal{F}_\ell^{(2)} \frac{G^2(1+F_{\ell,+})G^2(1-F_{\ell,-})}{G^2(1+\ell+F_{\ell,+})G^2(1-\ell-F_{\ell,-})}$$
$$\times \sum_{\substack{\{p\},\{h\}\\ n_p^{\pm}-n_h^{\pm}=\pm\ell}} \exp\left[\frac{2\pi i x}{L}\mathcal{P}_{ex}^{(d)}\right] \left(\frac{\sin(\pi F_{\ell,+})}{\pi}\right)^{2n_h^+} \left(\frac{\sin(\pi F_{\ell,-})}{\pi}\right)^{2n_h^-}$$
$$\times R_{n_p^+,n_h^+}(\{p^+\},\{h^+\}|F_{\ell,+})\; R_{n_p^-,n_h^-}(\{p^-\},\{h^-\}|-F_{\ell,-}),$$

The most amazing property of the critical sums is that these can be explicitly computed due to the following "magic" summation formula for $0 < x < L$ and $w = e^{2\pi i x/L-\varepsilon}$, with $0 < x < L$ (and taking the limit $\varepsilon \to +0$) :

Summation formula

$$f_\ell(\nu,w) = w^{\ell(\ell-1)/2}\frac{G^2(1+\ell+\nu)}{G^2(1+\nu)}(1-w)^{-(\nu+\ell)^2}$$

with $$f_\ell(\nu,w) \equiv \sum_{\substack{n_p,n_h=0\\ n_p-n_h=\ell}}^{\infty} \sum_{\substack{p_1<\cdots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_{n_h}\\ h_a\in\mathbb{N}^*}} w^{\sum_{j=1}^{n_p}(p_j-1)+\sum_{k=1}^{n_h} h_k}\left(\frac{\sin\pi\nu}{\pi}\right)^{2n_h}$$
$$\times \frac{\prod_{j>k}^{n_p}(p_j-p_k)^2\prod_{j>k}^{n_h}(h_j-h_k)^2}{\prod_{j=1}^{n_p}\prod_{k=1}^{n_h}(p_j+h_k-1)^2}\prod_{j=1}^{n_p}\frac{\Gamma^2(p_j+\nu)}{\Gamma^2(p_j)}\prod_{k=1}^{n_h}\frac{\Gamma^2(h_k-\nu)}{\Gamma 2(h_k)}$$

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$$\times \sum_{\substack{\{p\},\{h\}\\ n_p^{\pm}-n_h^{\pm}=\pm\ell}} \exp\left[\frac{2\pi i x}{L}\mathcal{P}_{ex}^{(d)}\right] \left(\frac{\sin(\pi F_{\ell,+})}{\pi}\right)^{2n_h^+} \left(\frac{\sin(\pi F_{\ell,-})}{\pi}\right)^{2n_h^-}$$

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$$f_\ell(\nu,w) = w^{\ell(\ell-1)/2}\frac{G^2(1+\ell+\nu)}{G^2(1+\nu)}(1-w)^{-(\nu+\ell)^2}$$

with $$f_\ell(\nu,w) \equiv \sum_{\substack{n_p,n_h=0\\ n_p-n_h=\ell}}^{\infty} \sum_{\substack{p_1<\dots<p_{n_p}\\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\dots<h_{n_h}\\ h_a\in\mathbb{N}^*}} w^{\sum_{j=1}^{n_p}(p_j-1)+\sum_{k=1}^{n_h}h_k} \left(\frac{\sin\pi\nu}{\pi}\right)^{2n_h}$$

$$\times \frac{\prod_{j>k}^{n_p}(p_j-p_k)^2\prod_{j>k}^{n_h}(h_j-h_k)^2}{\prod_{j=1}^{n_p}\prod_{k=1}^{n_h}(p_j+h_k-1)^2} \prod_{j=1}^{n_p}\frac{\Gamma^2(p_j+\nu)}{\Gamma^2(p_j)} \prod_{k=1}^{n_h}\frac{\Gamma^2(h_k-\nu)}{\Gamma 2(h_k)}$$

The two-point function asymptotic behavior

Using this identity we obtain for the two-point function

$$\langle \mathcal{O}_1(x')\, \mathcal{O}_2(x+x')\rangle_{cr} = \lim_{L\to\infty} \sum_{\ell=-\infty}^{\infty} \frac{e^{2ix\ell k_F}\, \overline{\mathcal{F}}_{\ell}^{(1)}\, \mathcal{F}_{\ell}^{(2)}\, L^{-\theta_\ell}}{\left(1-e^{\frac{2\pi ix}{L}}\right)^{(F_{\ell,+}+\ell)^2} \left(1-e^{-\frac{2\pi ix}{L}}\right)^{(F_{\ell,-}+\ell)^2}}$$

Now the thermodynamic limit can be easily taken and we obtain our final result

$$\langle \mathcal{O}_1(x')\, \mathcal{O}_2(x+x')\rangle_{cr} = \sum_{\ell=-\infty}^{\infty} \exp\left[2ix\ell k_F + i\frac{\pi}{2}\left((F_{\ell,-}+\ell)^2 - (F_{\ell,+}+\ell)^2\right)\right] \frac{\overline{\mathcal{F}}_{\ell}^{(1)}\, \mathcal{F}_{\ell}^{(2)}}{(2\pi x)^{\theta_\ell}}$$

This is an explicit result for the contribution of the critical form factors to the two-point functions of a very large class of integrable critical models. Our main conjecture states that this equation gives the leading asymptotic behavior of all the oscillation harmonics of the two-point function.

Perspectives

- This scheme applies to the XXZ Heisenberg chain (Véronique talk) and to the Bose gas, even there for time dependent case (Karol talk).
- It applies to a large class of integrable models as soon as the form factors are known and posses the so-called kinematical poles between particles and holes (or particles and their corresponding anti-particles).
- I would like to further argue that this scheme applies even to non-integrable critical cases provided some minimal assumptions are satisfied; hence it should give a microscopic (particle like) description of CFT models. It is interesting to notice the origin of (non-integer) critical exponents in this framework : the polarization of the physical vacuum (that encodes the effect of interactions) when creating particle-hole excitations at the Fermi boundaries. The "magic" summation formula then works in these cases these polarizations at the right and left Fermi boundaries entering as given (physical) parameters through the shift function values.