Numerical analysis of String solutions of the Integrable XXZ spin chains

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Plan of my Talk

1. Introduction (XXZ model, Bethe ansatz, string hypothesis)
2. s=1/2 XXX chain (Review)
3. s=1/2 massive XXZ chain (Original)
4. s=1 XXX chain (Original)
5. s=1 massive XXZ chain (Original, cf. Talk by Motegi)

Spin-1/2 Heisenberg XXZ Chain

$$\mathcal{H} = \sum_{n=1}^{N} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z\right]$$

$-1 < \Delta \leq 1$ **: massless region**

$\Delta > 1$ **: massive region**

$\Delta = 1$ **: Anti-ferromagnetic XXX chain**

Exactly diagonalized by Bethe ansatz

Bethe (1931),
Orbach (1958),
Walker (1959), …

Bethe Roots

$$\{\lambda_1, \lambda_2, \cdots, \lambda_M\}$$

Bethe Ansatz equation for s=1/2 XXZ chain

$$\left(\frac{\varphi(\lambda_j + \mathrm{i}\zeta/2)}{\varphi(\lambda_j - \mathrm{i}\zeta/2)}\right)^N = \prod_{\ell \neq j}^{M} \frac{\varphi(\lambda_j - \lambda_\ell + \mathrm{i}\zeta)}{\varphi(\lambda_j - \lambda_\ell - \mathrm{i}\zeta)}$$

$$\begin{cases} \varphi(\lambda) = \lambda, & \zeta = 1 & \text{XXX case} \\ \varphi(\lambda) = \sinh\lambda, & \Delta = \cos\zeta & \text{massless case} \\ \varphi(\lambda) = \sin\lambda, & \Delta = \cosh\zeta & \text{massive case} \end{cases}$$

Generalization to spin-s case

$$\left(\frac{\varphi(\lambda_j + \mathrm{i}s\zeta)}{\varphi(\lambda_j - \mathrm{i}s\zeta)}\right)^N = \prod_{\ell \neq j}^{M} \frac{\varphi(\lambda_j - \lambda_\ell + \mathrm{i}\zeta)}{\varphi(\lambda_j - \lambda_\ell - \mathrm{i}\zeta)}$$

A.B. Zamolodchikov -Fateev (1980)
Kulish-Reshetikhin-Sklyanin (1981)
Babujian (1982)
Sogo-Akutsu-Abe (1983)
Babujian-Tsvelick (1986)
Kirillov-Reshetikhin (1987)
Deguchi-Wadati-Akutsu (1988)

Bethe Eigenstate

$$B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|0\rangle$$

Logarithmic form of Bethe ansatz equation

$$\theta_{2s}(\lambda_j) = \frac{2\pi}{N} I_j + \frac{1}{N} \sum_{\ell \neq j}^{M} \theta_2(\lambda_j - \lambda_\ell), \quad (\mathrm{mod}\ 2\pi),$$

Bethe quantum number: $I_j = (N - M + 1)/2 \quad (\mathrm{mod}\ 1)$

$$\begin{cases} \theta_n(\lambda) = 2\tan^{-1}\left(\dfrac{2\lambda}{n}\right) & \text{XXX case} \\ \theta_n(\lambda) = 2\tan^{-1}\left(\dfrac{\tanh\lambda}{\tan(n\zeta/2)}\right) & \text{massless case} \\ \theta_n(\lambda) = 2\tan^{-1}\left(\dfrac{\tan\lambda}{\tanh(n\zeta/2)}\right) + 2\pi\nu(\lambda) & \text{massive case} \end{cases}$$

$$\nu(\lambda) = \begin{cases} 1 & \pi/2 < \mathrm{Re}\lambda < \pi \\ 0 & -\pi/2 < \mathrm{Re}\lambda < \pi/2 \\ -1 & -\pi < \mathrm{Re}\lambda < -\pi/2 \end{cases}$$

Self-conjugacy: $\{\lambda_1^*, \lambda_2^*, \cdots, \lambda_M^*\} = \{\lambda_1, \lambda_2, \cdots, \lambda_M\}$ Vladimirov (1986)

Bethe ansatz equation: $$\left(\frac{\varphi(\lambda_j + \mathrm{i}s\zeta)}{\varphi(\lambda_j - \mathrm{i}s\zeta)}\right)^N = \prod_{\ell \neq j}^{M} \frac{\varphi(\lambda_j - \lambda_\ell + \mathrm{i}\zeta)}{\varphi(\lambda_j - \lambda_\ell - \mathrm{i}\zeta)}$$

If Imλ>0, then $\left|\frac{\varphi(\lambda_j + \mathrm{i}s\zeta)}{\varphi(\lambda_j - \mathrm{i}s\zeta)}\right| > 1$ **and LHS → ∞ (N → ∞)**

➡ **This implies that** $\lambda_j - {}^{\exists}\lambda_k = \mathrm{i}\zeta + \delta_{jk}, \quad |\delta_{jk}| \ll 1$

n-string: $\left\{\lambda + \frac{\mathrm{i}\zeta}{2}(n-1) + \delta_1,\ \lambda + \frac{\mathrm{i}\zeta}{2}(n-3) + \delta_2, \cdots,\ \lambda - \frac{\mathrm{i}\zeta}{2}(n-1) + \delta_1^*\right\}$

$\lambda \in \mathbb{R}$, λ **: string center**

δ_j **: string deviation**

Bethe (1931), Takahashi (1971), Gaudin (1971), Takahashi-Suzuki (1972), …

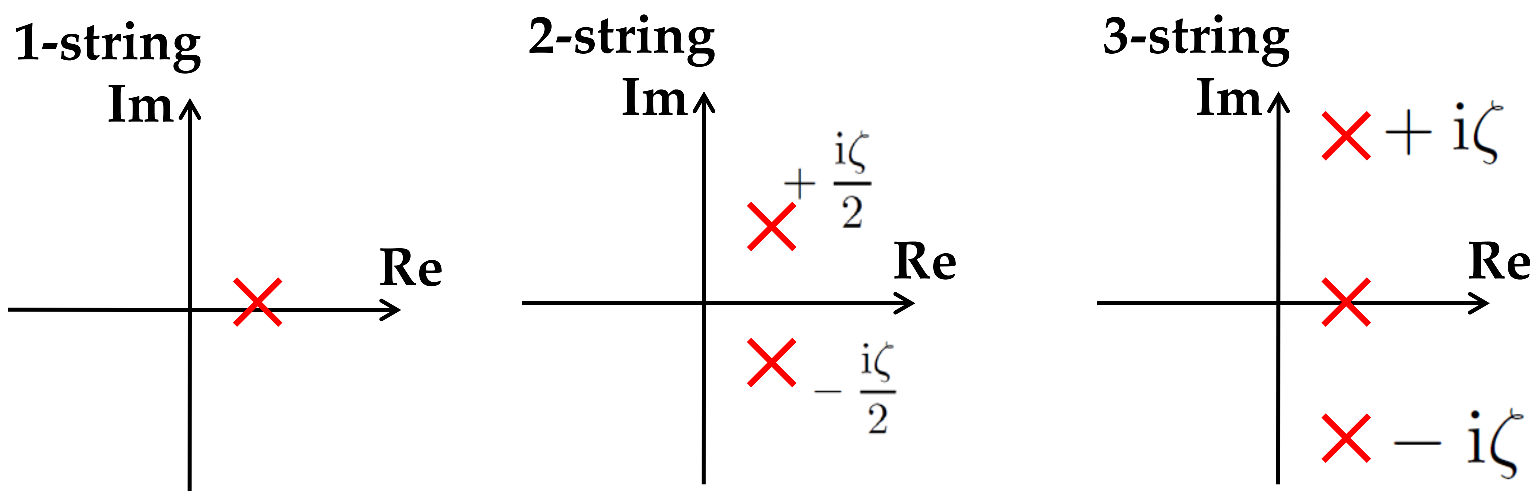
String Hypothesis: $\delta_j \to 0 \quad (N \to \infty)$

Distribution of Bethe Roots in the complex plain

n-string: $\left\{\lambda+\frac{i\zeta}{2}(n-1)+\delta_1,\ \lambda+\frac{i\zeta}{2}(n-3)+\delta_2,\cdots,\ \lambda-\frac{i\zeta}{2}(n-1)+\delta_1^*\right\}$

$\lambda\in\mathbb{R}$, λ **: string center**

δ_j **: string deviation**



If we assume string deviations vanish,
we obtain Bethe-Takahashi equation for string centers:

$$\sum_{\ell=1}^{2s} \theta_{2s-2\ell+n+1}(\lambda_j^{(n)}) = 2\pi \frac{I_j^{(n)}}{N} + \frac{1}{N} \sum_{m,k} \Theta_{nm}(\lambda_j^{(n)} - \lambda_k^{(m)})$$

$\lambda_j^{(n)}$: **String center of *j*-th *n*-string**

$$\Theta_{nm} = 2(1-\delta_{nm})\theta_{|n-m|} + 4\theta_{|n-m|+2} + \cdots + 4\theta_{n+m-2} + 2\theta_{n+m}$$

1. **Completeness of Bethe states**
2. **Thermodynamic Bethe ansatz**

Bethe (1931), Takahashi (1971), Gaudin (1971), Takahashi-Suzuki (1972), …

However,

1. **Counterexamples of string hypothesis (cf. s=1/2, XXX case)**
2. **It is very elusive to determine string deviations numerically**

Counterexamples of string hypothesis:

s=1/2, XXX case

A.A. Vladimirov, Phys. Lett. A 105 418 (1984),
F.H.L Essler, V.E. Korepin, K. Schoutens, J. Phys. A: Math. Gen. 25 4115 (1992),
etc, …

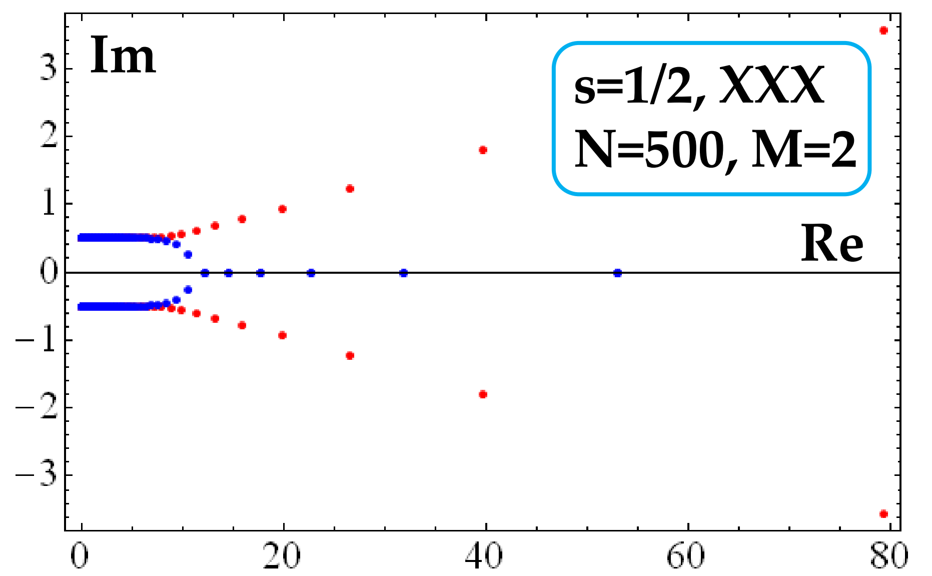
s=1/2, massless case

A. Ilakovac, M. Kolanovic, S. Pallua, P. Prester, Phys. Rev. B 60 7271 (1999),
etc, …

Novel method to determine string deviations correctly for s=1/2, XXX case
R. Hagemans, J-S. Caux, J. Phys. A: Math. Theor. 40 14605 (2007)

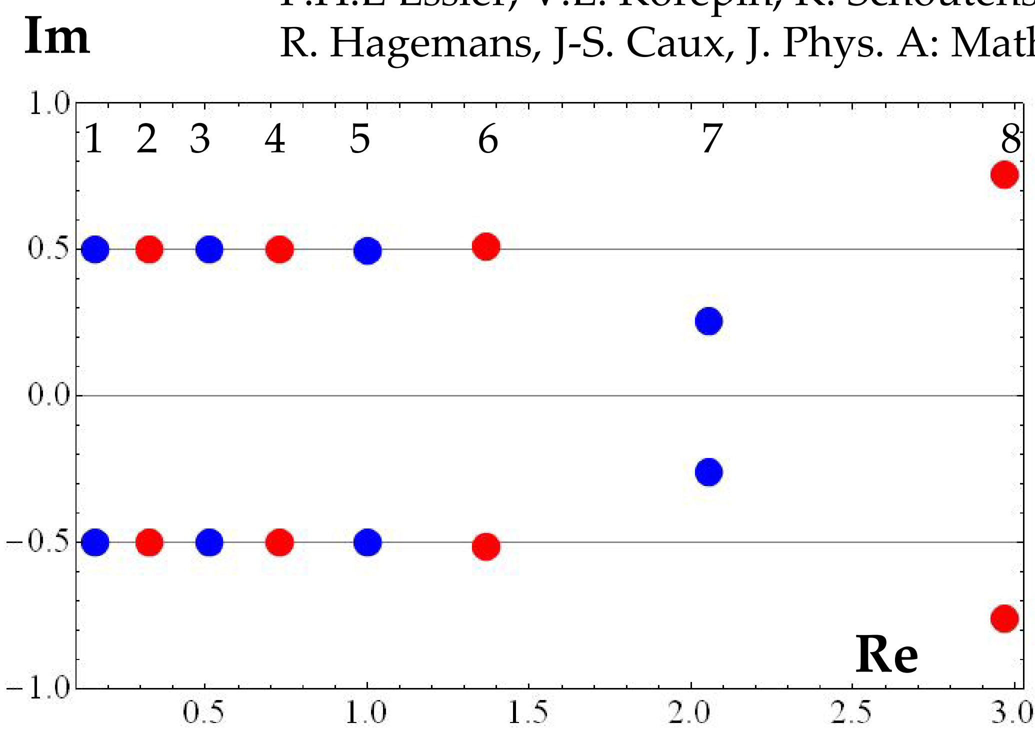
We generalize this method to

1. **s=1/2 XXZ chain**
2. **Integrable s=1 XXZ chain**



2-string for s=1/2 XXX chain

A.A. Vladimirov, Phys. Lett. A 105 418 (1984),
F.H.L Essler, V.E. Korepin, K. Schoutens, J. Phys. A: Math. Gen. 25 4115 (1992),
R. Hagemans, J-S. Caux, J. Phys. A: Math. Theor. 40 14605 (2007), etc

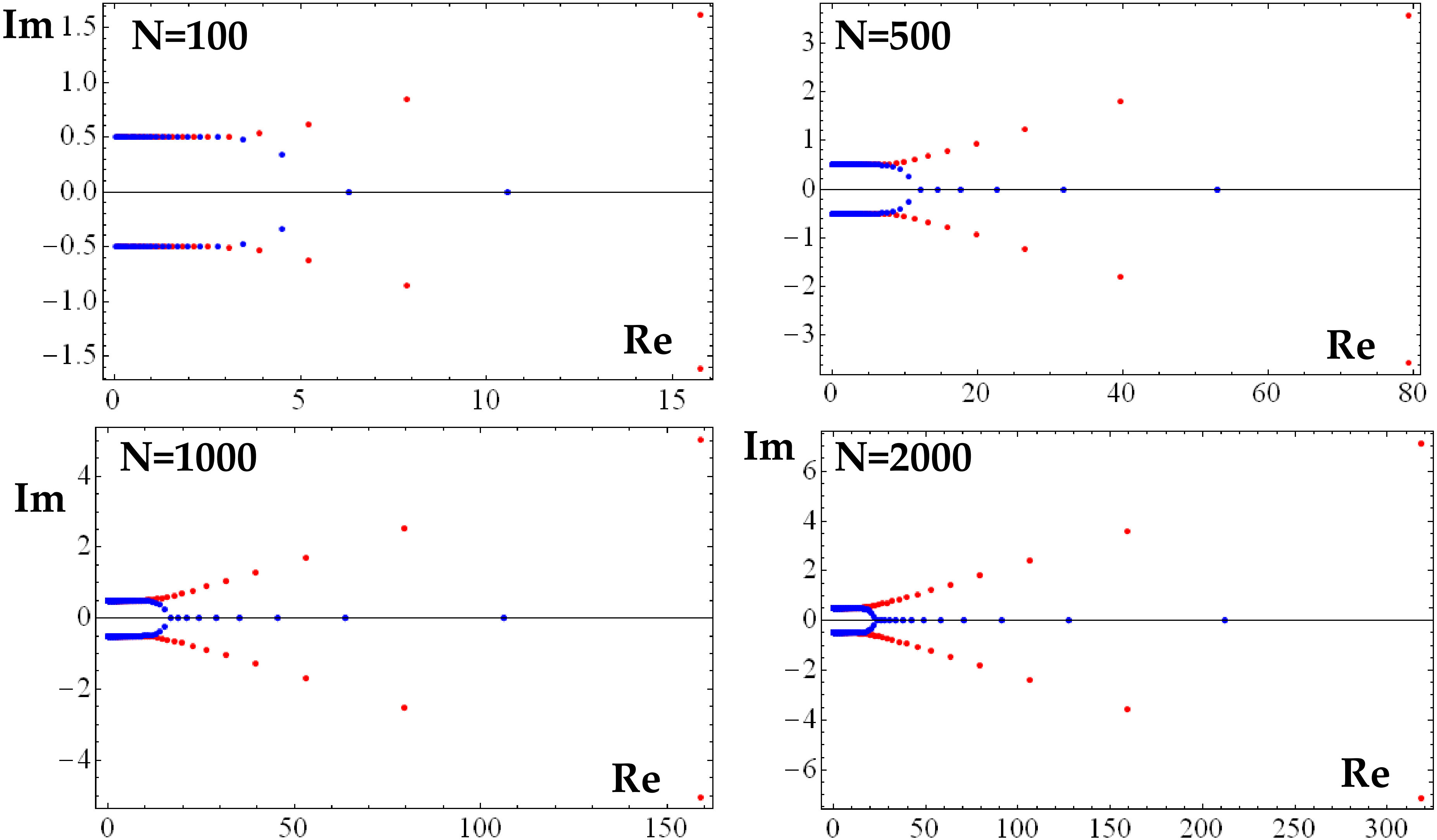


N=20, M=2, 2-string ⇒ 8 states

Red: Wide string |Im λ|>1/2
Blue: Narrow string |Im λ|<1/2

1. $\lambda =$ 0.15838444032453694 ± 0.49999999999999994 i
2. $\lambda =$ 0.32491969603887527 ± 0.50000000006304450 i
3. $\lambda =$ 0.50952572099684120 ± 0.49999986166251836 i
4. $\lambda =$ 0.72650920415133000 ± 0.50002420950057140 i
5. $\lambda =$ 1.0009931965219054 ± 0.49900482666880420 i
6. $\lambda =$ 1.3669339403944727 ± 0.51275214737240580 i
7. $\lambda =$ 2.0522067441202110 ± 0.25715752928188960 i
8. $\lambda =$ 2.9675855287321000 ± 0.75942429286804190 i

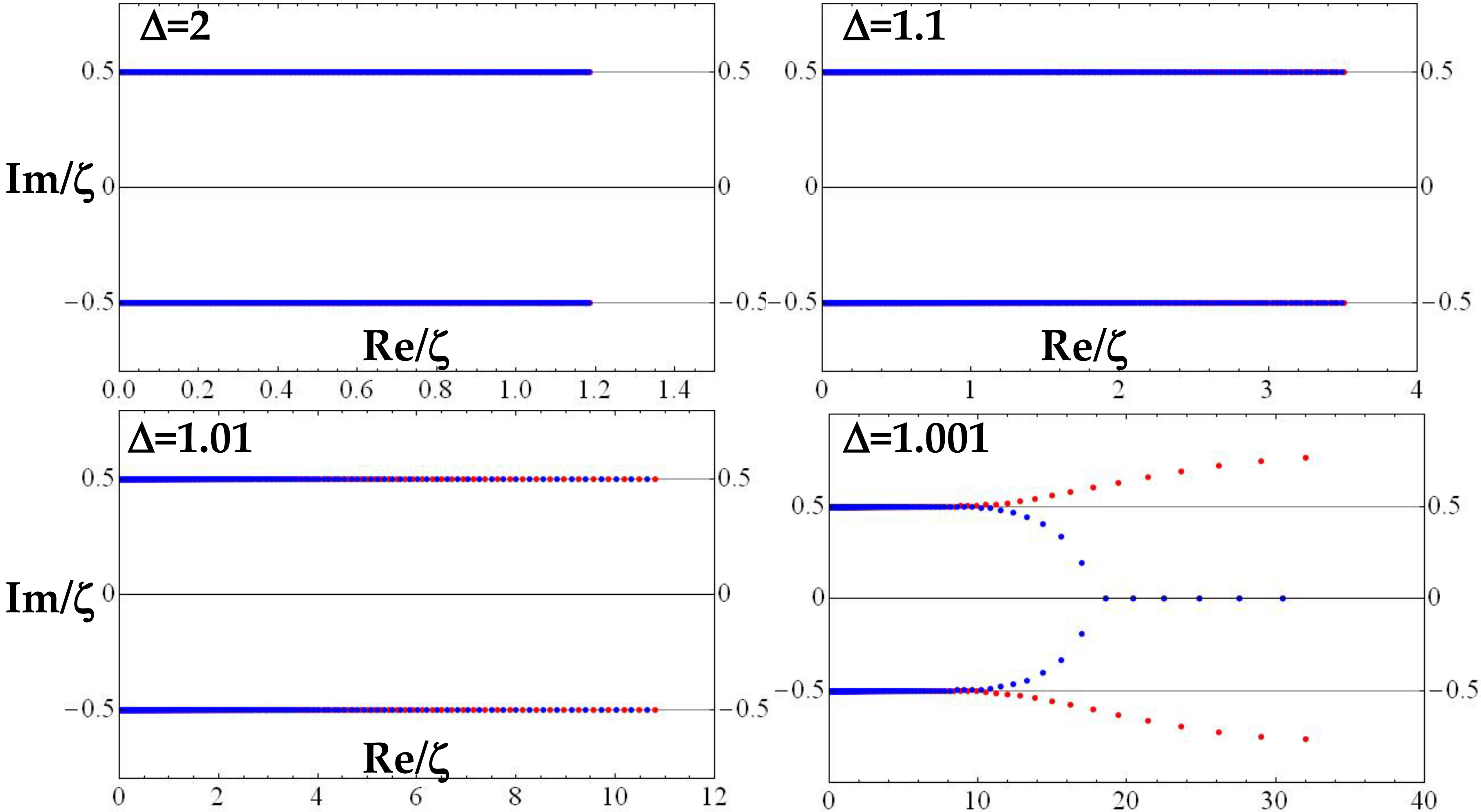
Violation of 2-string for s=1/2 XXX chain



1. Imaginary part of wide strings diverges.
2. Narrow strings eventually fall onto the real axis.

2-string for s=1/2 massive XXZ chain

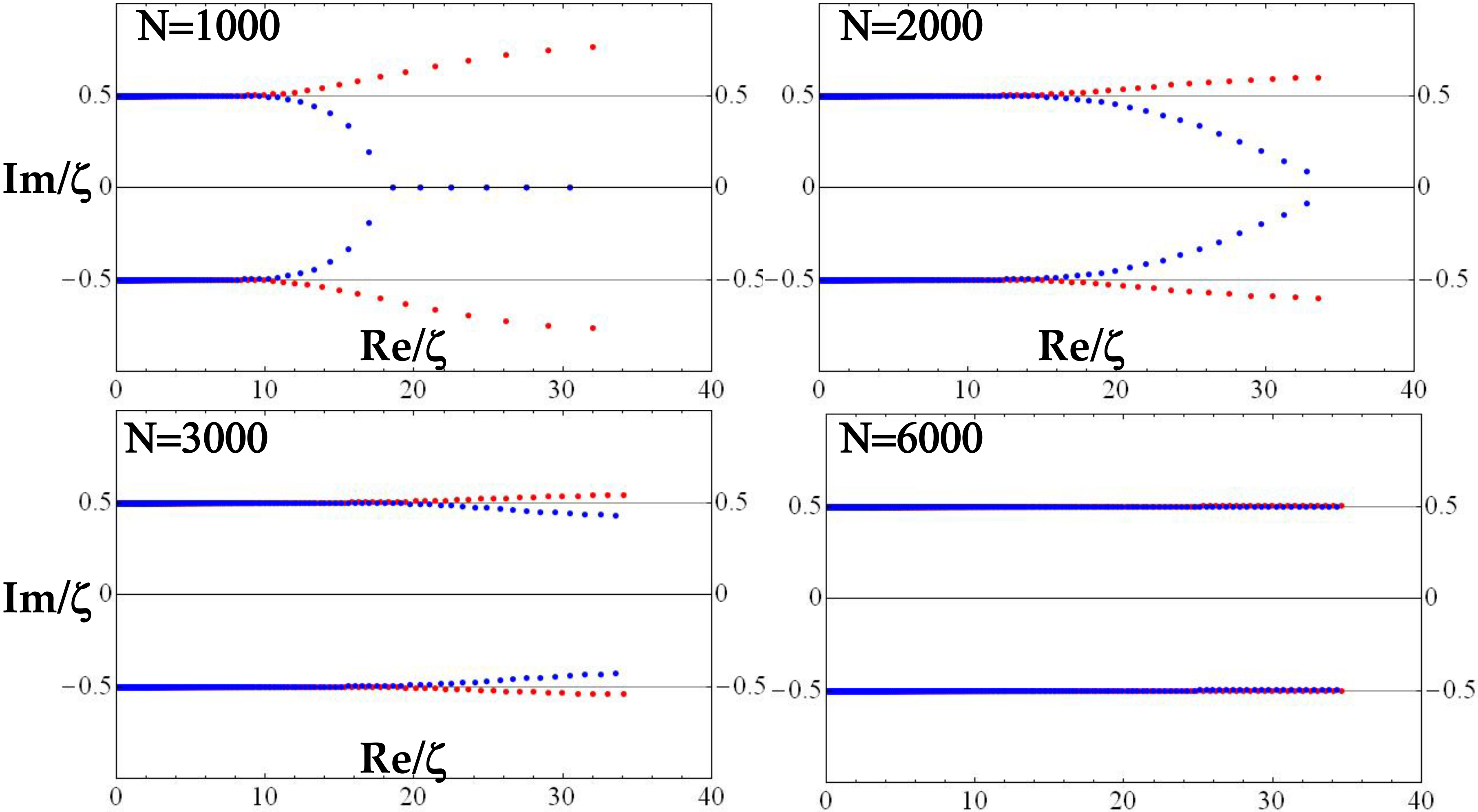
JS-Deguchi (2011)



1. Decrease Δ close to 1 with fixed system size N=1000.
2. All 2-strings for large Δ are well described by the string hypothesis.
3. When Δ approaches to 1, string hypothesis begins to break down.

2-string for s=1/2 massive XXZ chain

JS-Deguchi (2011)



1. Increase system size with fixed anisotropy Δ =1.001.
2. String configurations recovers for large N.
3. We can conclude that all 2-strings in the massive XXZ chain are well described by the string hypothesis

Integrable spin-1 XXZ chain

1. It has been argued that the ground state is given by N/2 set of 2-strings.

   (A. Klümper, M. Batchelor and P.A. Pearce, J. Phys. A: Math. Gen. 24 (1991) 3111–3133)

2. In the thermodynamic limit, string deviations diverge or not? (String hypothesis breaks down as in the spin-1/2 case?)

↓

1. Comparison with the exact diagonalization for N=8.
2. We calculate exact Bethe roots (without assuming string hypothesis) up to N=10000.

Confirmation of the ground state of Integrable spin-1 XXX chain

Exact diagonalization

E= -1.020085616651852,

-0.968149353581999,

-0.919044154482835 2,

-0.905892155774171 7,

-0.868456934807929 4,........

System size: N=8

Energy of the N/2 set of 2-strings

(Exact Bethe roots without assuming string hypothesis)

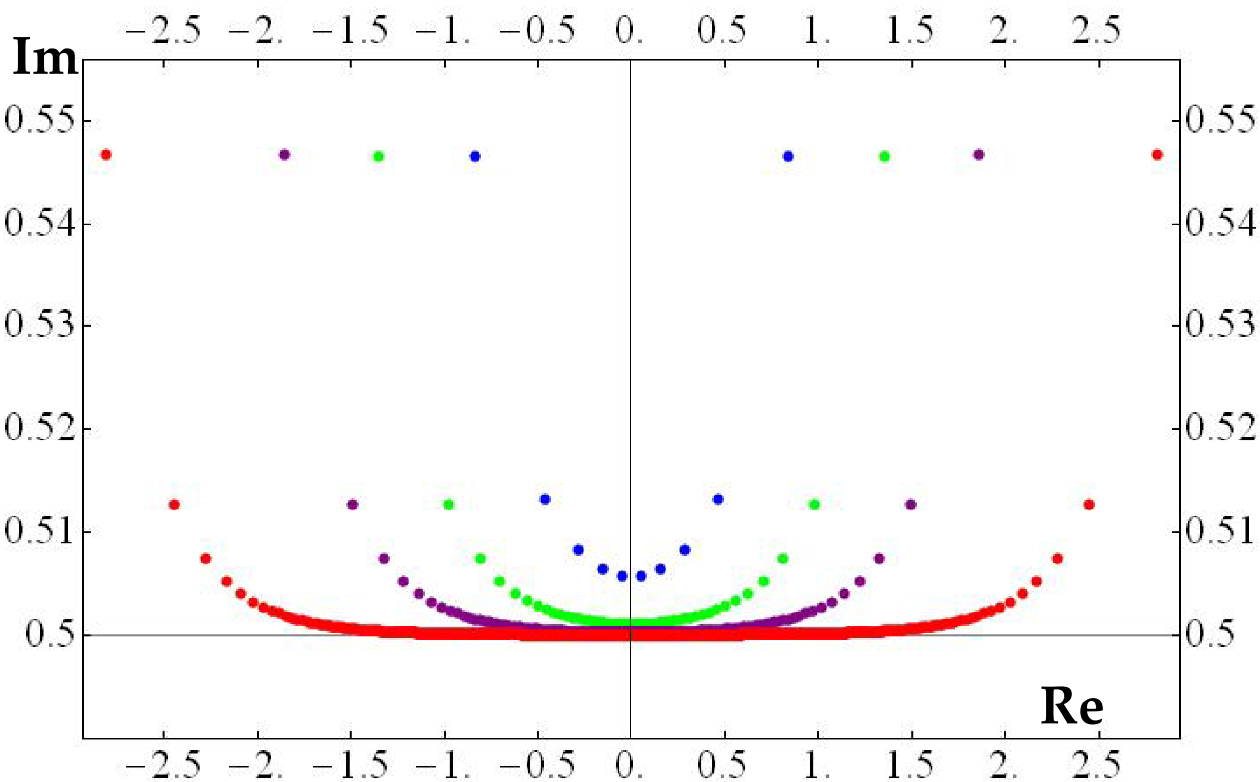
E = -1. 020085616651854

cf. Energy of the N/2 set of 2-strings

(Bethe-Takahashi equation)

E = -1.013350454279014 2

Bethe roots for the ground state of integrable spin-1 XXX chain



Blue: N = 20
Green: N = 100
Purple: N = 500
Red: N = 10000

1. The largest string deviation is almost the same for 20 < N < 10000
2. String deviation remains finite in the thermodynamic limit.

Confirmation of the ground state of Integrable spin-1 massive XXZ chain

Exact diagonalization

E= -4.035763938458895,
-3.994242289320799,
-3.941525866578427 6,
-3.703991384220407 7,
-3.635511588761259 5,........

Δ=2,
System size: N=8

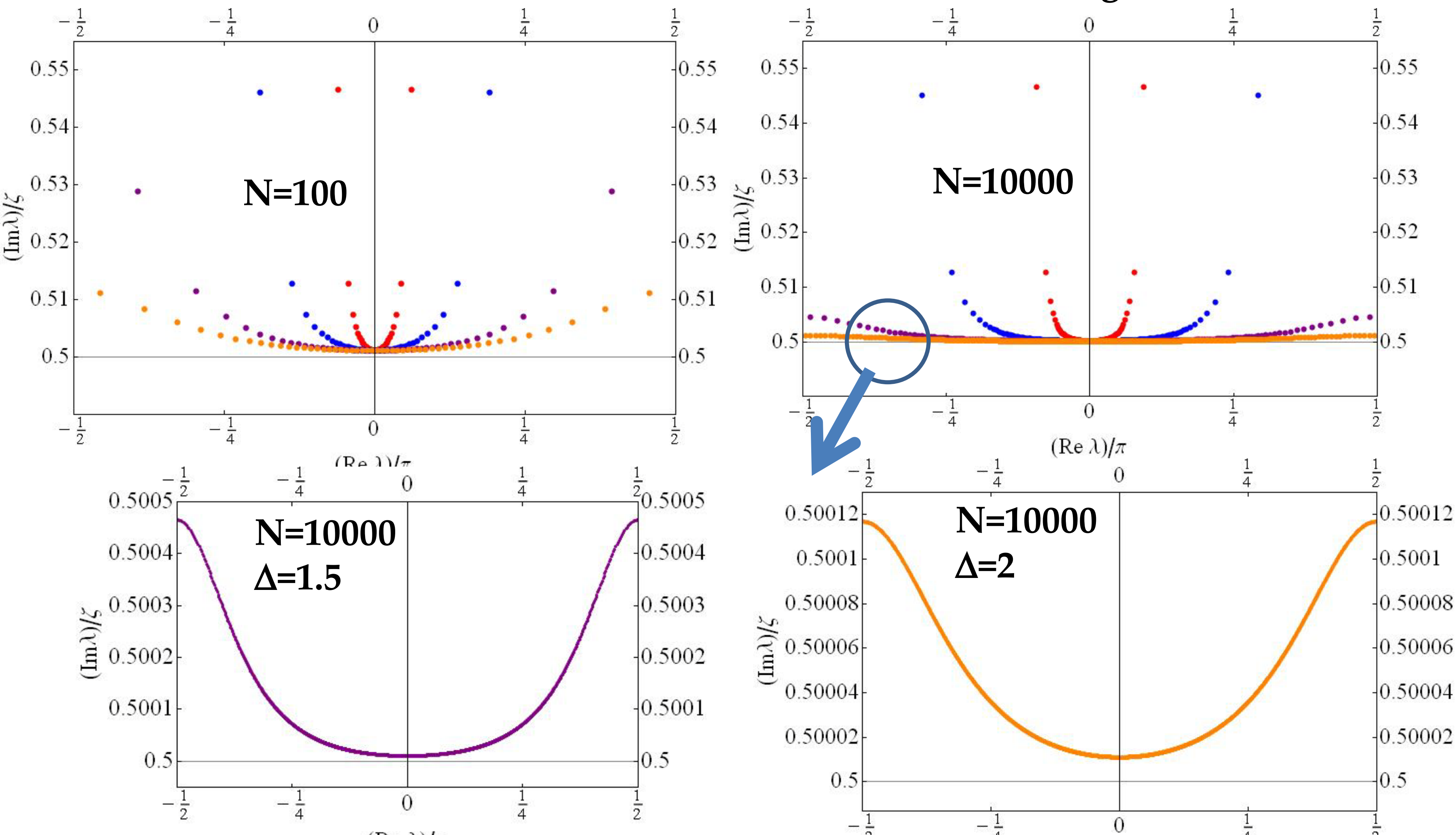
Energy of the N/2 set of 2-strings
(Exact Bethe roots without assuming string hypothesis)
E = - 4.035763938458905

cf. Energy of the N/2 set of 2-strings
(Bethe-Takahashi equation)
E = -4.02454623769426 5

cf. Talk by K. Motegi

Bethe roots for the ground state of integrable spin-1 massive XXZ chain

Red: $\Delta = 1.01$
Blue: $\Delta = 1.1$
Purple: $\Delta = 1.5$
Orange: $\Delta = 2$



String deviation remains finite in the thermodynamic limit.

Summary

We developed a method to obtain correct Bethe roots including the string deviations for s=1/2 and s=1 XXZ chain.

s=1/2 massive XXZ chain

1. All 2-strings for large Δ are well described by the string hypothesis.
2. When Δ approaches to 1, string hypothesis begins to break down.
3. However, we observe that string hypothesis recovers with increasing N.

s=1 XXX and massive XXZ chain

1. We have confirmed that N/2 set of 2-strings gives the ground state.
2. We have observed that the string deviations remain finite in the thermodynamic limit.