Scalar products in models with an $sl(3)$ R-matrix

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Outline

- Review of algebraic Bethe Ansatz for the $sl(2)$ and $sl(3)$ XXX spin-chains.
- A set of uniquely-determining properties of the $sl(2)$ XXX scalar product.
- Determinant solution of the preceding conditions.
- A set of uniquely-determining properties of the $sl(3)$ XXX scalar product.
- A partial result towards solving these conditions.

Algebraic Bethe Ansatz for models with $sl(2)$ symmetry

- We define $a(u) = u + 1,\ b(u) = u,\ c(u) = 1.$ The R-matrix is

$$R_{\alpha\beta}(u) = \left(\begin{array}{cc|cc} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ \hline 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{array} \right)_{\alpha\beta}$$

- The monodromy matrix is

$$T_\alpha(u) = \left(\begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array} \right)_\alpha$$

and satisfies the intertwining relation

$$R_{\alpha\beta}(u-v)T_\alpha(u)T_\beta(v) = T_\beta(v)T_\alpha(u)R_{\alpha\beta}(u-v)$$

Algebraic Bethe Ansatz for models with $sl(2)$ symmetry

- The transfer matrix is $\mathcal{T}(u) = A(u) + D(u)$. Eigenvectors of $\mathcal{T}(u)$ are given by

$$|v_1, \ldots, v_m\rangle = B(v_1) \ldots B(v_m)|0\rangle$$

$$\langle v_m, \ldots, v_1| = \langle 0|C(v_m) \ldots C(v_1)$$

  with $\{v_1, \ldots, v_m\}$ satisfying the Bethe equations.

- The scalar product $S(\{u\}_m|\{v\}_m)$ is defined as

$$S(\{u\}_m|\{v\}_m) = \langle u_m, \ldots, u_1|v_1, \ldots, v_m\rangle$$

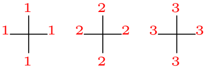
Nested Bethe Ansatz for models with $sl(3)$ symmetry

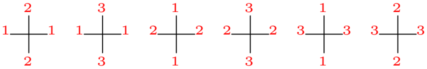
- The R-matrix is

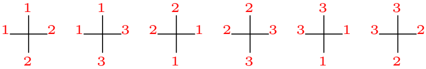
$$R^{(1)}_{\alpha\beta}(u) = \left(\begin{array}{ccc|ccc|ccc} a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(u) & 0 & c(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(u) & 0 & 0 & 0 & c(u) & 0 & 0 \\ \hline 0 & c(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(u) & 0 & c(u) & 0 \\ \hline 0 & 0 & c(u) & 0 & 0 & 0 & b(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c(u) & 0 & b(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(u) \end{array}\right)_{\alpha\beta}$$

and we define $R^{(2)}_{\alpha\beta}(u) \equiv R_{\alpha\beta}(u)$.

- Represent the entries as follows:

a: 

b: 

c: 

Nested Bethe Ansatz for models with $sl(3)$ symmetry

- The monodromy matrix is

$$T^{(1)}_{\alpha}(u) = \begin{pmatrix} t_{11}(u) & t_{12}(u) & t_{13}(u) \\ t_{21}(u) & t_{22}(u) & t_{23}(u) \\ t_{31}(u) & t_{32}(u) & t_{33}(u) \end{pmatrix}_{\alpha} = \begin{pmatrix} A^{(1)}(u) & B^{(1)}_{\beta}(u) \\ C^{(1)}_{\gamma}(u) & D^{(1)}_{\delta}(u) \end{pmatrix}$$

where we have defined

$$B^{(1)}_{\beta}(u) = \begin{pmatrix} t_{12}(u) & t_{13}(u) \end{pmatrix}_{\beta} \qquad D^{(1)}_{\delta}(u) = \begin{pmatrix} t_{22}(u) & t_{23}(u) \\ t_{32}(u) & t_{33}(u) \end{pmatrix}_{\delta}$$

- The intertwining equation is

$$R^{(1)}_{\alpha\beta}(u-v)T^{(1)}_{\alpha}(u)T^{(1)}_{\beta}(v) = T^{(1)}_{\beta}(v)T^{(1)}_{\alpha}(u)R^{(1)}_{\alpha\beta}(u-v)$$

Nested Bethe Ansatz for models with $sl(3)$ symmetry

- We also need

$$T^{(2)}_{\alpha}(u|v_m,\dots,v_1) = D^{(1)}_{\alpha}(u) R^{(2)}_{\alpha\alpha_m}(u-v_m)\dots R^{(2)}_{\alpha\alpha_1}(u-v_1)$$

$$\equiv \begin{pmatrix} A^{(2)}(u|\{v\}_m) & B^{(2)}(u|\{v\}_m) \\ C^{(2)}(u|\{v\}_m) & D^{(2)}(u|\{v\}_m) \end{pmatrix}_{\alpha}$$

and

$$T^{(2)}_{\alpha}(v_1,\dots,v_m|u) = R^{(2)}_{\alpha\alpha_1}(u-v_1)\dots R^{(2)}_{\alpha\alpha_m}(u-v_m) D^{(1)}_{\alpha}(u)$$

$$\equiv \begin{pmatrix} A^{(2)}(\{v\}_m|u) & B^{(2)}(\{v\}_m|u) \\ C^{(2)}(\{v\}_m|u) & D^{(2)}(\{v\}_m|u) \end{pmatrix}_{\alpha}$$

Nested Bethe Ansatz for models with $sl(3)$ symmetry

- The transfer matrix is $\mathcal{T}(u) = t_{11}(u) + t_{22}(u) + t_{33}(u)$. Eigenvectors of $\mathcal{T}(u)$ are given by

  $$|\{v\}_m, \{u\}_n\rangle = B^{(1)}_{\alpha_1}(v_1) \dots B^{(1)}_{\alpha_m}(v_m) B^{(2)}(u_1|\{v\}_m) \dots B^{(2)}(u_n|\{v\}_m)|0\rangle \otimes |2^m\rangle_\alpha$$

  $$\langle\{u\}_n, \{v\}_m| = \langle 2^m|_\alpha \otimes \langle 0|C^{(2)}(\{v\}_m|u_n) \dots C^{(2)}(\{v\}_m|u_1) C^{(1)}_{\alpha_m}(v_m) \dots C^{(1)}_{\alpha_1}(v_1)$$

  with $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ satisfying the nested Bethe equations.

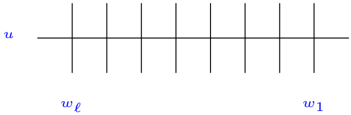
- The scalar product $S(\{\mu\}_n, \{\nu\}_m|\{v\}_m, \{u\}_n)$ is defined as

  $$S(\{\mu\}_n, \{\nu\}_m|\{v\}_m, \{u\}_n) = \langle\{\mu\}_n, \{\nu\}_m|\{v\}_m, \{u\}_n\rangle$$

Specialization to XXX model — $sl(2)$ case

- Consider an XXX spin-chain of length ℓ. The monodromy matrix becomes

$$T_\alpha(u) = R_{\alpha\ell}(u - w_\ell)\dots R_{\alpha 1}(u - w_1)$$



- The pseudo-vacuum states $|0\rangle$ and $\langle 0|$ are chosen to be

$$|0\rangle = |1^\ell\rangle \equiv \prod_{i=1}^{\ell} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \qquad \langle 0| = \langle 1^\ell| \equiv \prod_{i=1}^{\ell} \begin{pmatrix} 1 & 0 \end{pmatrix}_i$$

- The Bethe equations for this model:

$$\prod_{j=1}^{\ell} \left(\frac{v_i - w_j + 1}{v_i - w_j} \right) = \prod_{\substack{j \neq i \\ j=1}}^{m} \left(\frac{v_i - v_j + 1}{v_i - v_j - 1} \right)$$

Specialization to XXX model — $sl(3)$ case

- Consider an XXX spin-chain of length ℓ. The monodromy matrix becomes

$$T_\alpha^{(1)}(u) = R_{\alpha\ell}^{(1)}(u - w_\ell) \dots R_{\alpha 1}^{(1)}(u - w_1)$$

- The pseudo-vacuum states $|0\rangle$ and $\langle 0|$ are chosen to be

$$|0\rangle = |1^\ell\rangle \equiv \prod_{i=1}^{\ell} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_i \qquad \langle 0| = \langle 1^\ell| \equiv \prod_{i=1}^{\ell} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}_i$$

- The two sets of Bethe equations for this model:

$$\prod_{j=1}^{\ell} \left(\frac{v_i - w_j + 1}{v_i - w_j} \right) \prod_{k=1}^{n} \left(\frac{v_i - u_k}{v_i - u_k - 1} \right) = \prod_{\substack{j \neq i \\ j=1}}^{m} \left(\frac{v_i - v_j + 1}{v_i - v_j - 1} \right)$$

$$\prod_{j=1}^{m} \left(\frac{u_i - v_j + 1}{u_i - v_j} \right) = \prod_{\substack{j \neq i \\ j=1}}^{n} \left(\frac{u_i - u_j + 1}{u_i - u_j - 1} \right)$$

Properties of $sl(2)$ XXX scalar product

- We define the *intermediate* scalar products

$$S(\{u\}_n|\{v\}_m) = \langle 1^{\ell-m+n}, 2^{m-n}|C(u_n)\dots C(u_1)B(v_1)\dots B(v_m)|1^\ell\rangle$$

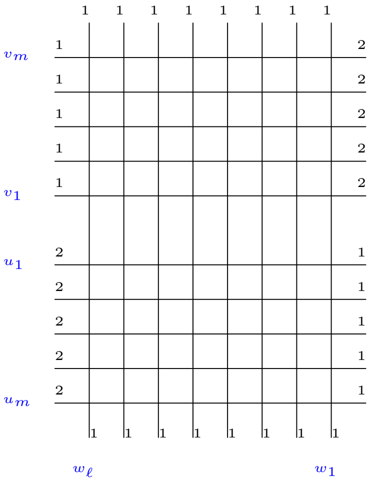
- They satisfy a set of uniquely-determining conditions. $S(\{u\}_n|\{v\}_m)$
  1. Is a polynomial in u_n of degree $\ell-1$, with zeros at $u_n = w_i$ for all $1 \le i \le m-n$.
  2. Is symmetric in the variables $\{w_{m-n+1}, \dots, w_\ell\}$.
  3. Satisfies the recursion relation

$$S(\{u\}_n|\{v\}_m)\Big|_{u_n+1=w_{m-n+1}} = -\prod_{i=1}^{\ell}(w_{m-n+1} - w_i - 1)S(\{u\}_{n-1}|\{v\}_m)$$

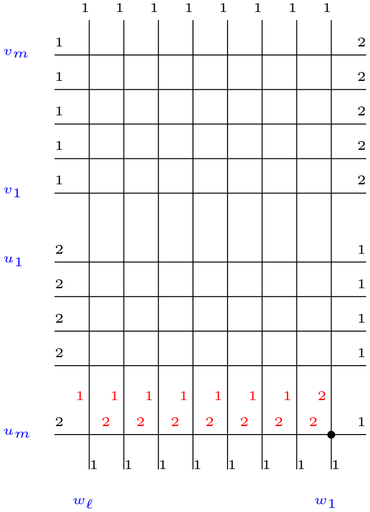
  4. $S(\{u\}_0|\{v\}_m) = \prod_{i=1}^m \prod_{j=m+1}^{\ell}(v_i - w_j + 1)Z(\{v\}_m|\{w\}_m)$.

Recursion procedure – $sl(2)$ scalar product

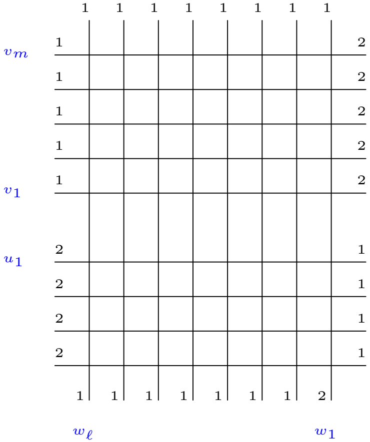
$$\langle 1^{\ell} | C(u_m) \dots C(u_1) B(v_1) \dots B(v_m) | 1^{\ell} \rangle$$



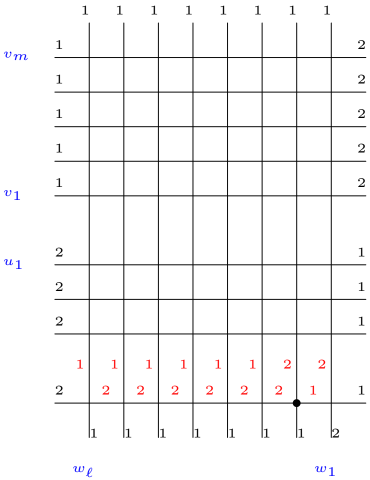
Recursion procedure – $sl(2)$ scalar product



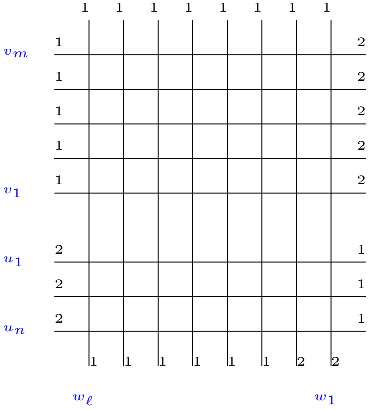
Recursion procedure – $sl(2)$ scalar product



Recursion procedure – $sl(2)$ scalar product

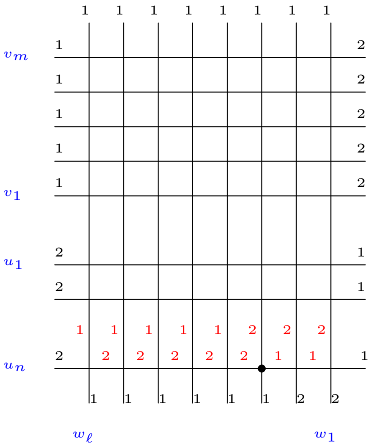


Recursion procedure – $sl(2)$ scalar product

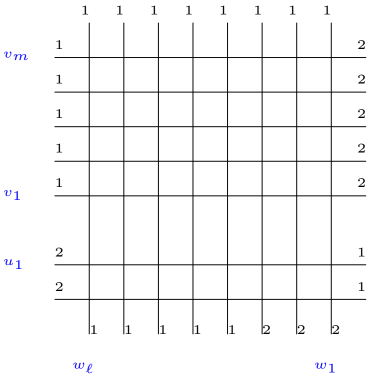


$$\langle 1^{\ell-m+n}, 2^{m-n} | C(u_n) \dots C(u_1) B(v_1) \dots B(v_m) | 1^\ell \rangle$$

Recursion procedure – $sl(2)$ scalar product



Recursion procedure – $sl(2)$ scalar product



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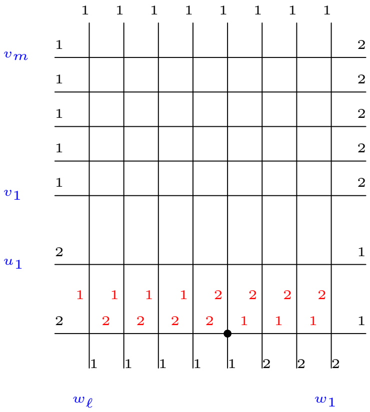

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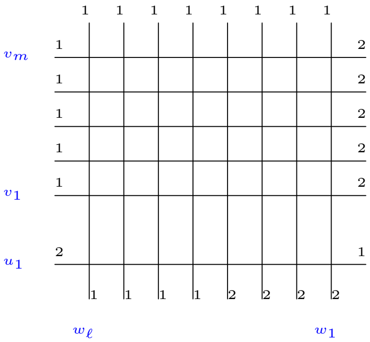

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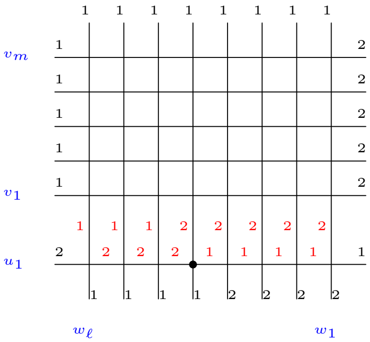
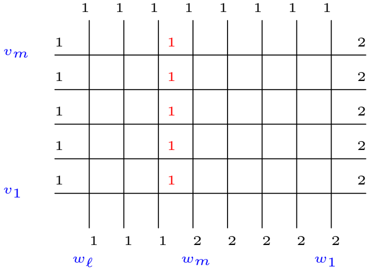
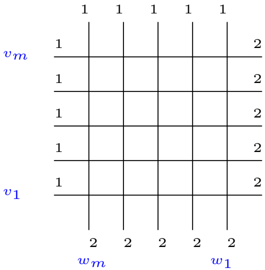


End of recursion – $sl(2)$ scalar product



$$\langle 1^{\ell-m}, 2^m | B(v_1) \dots B(v_m) | 1^\ell \rangle$$

Domain wall partition function



- $Z(\{v\}_m|\{w\}_m)$ is the domain wall partition function, given by the Izergin/Korepin formula

$$Z(\{v\}_m|\{w\}_m) = \frac{\prod_{i,j=1}^{m}(v_i - w_j + 1)(v_i - w_j)}{\prod_{1\le i<j\le m}(v_i - v_j)(w_j - w_i)} \times \det\left(\frac{1}{(v_i - w_j + 1)(v_i - w_j)}\right)_{1\le i,j\le m}$$

Solution of conditions

- Define the functions

$$f_i(w) = \frac{1}{(v_i - w + 1)} \prod_{k \neq i}^{m} (v_k - w)$$

$$g_i(u) = \frac{1}{(v_i - u)} \left(\prod_{k \neq i}^{m} (v_k - u + 1) \prod_{k=1}^{\ell} (u - w_k + 1) - \prod_{k \neq i}^{m} (v_k - u - 1) \prod_{k=1}^{\ell} (u - w_k) \right)$$

- Using these construct the $m \times m$ matrix

$$\mathcal{M}(\{u\}_n | \{v\}_m) = \left(\begin{array}{ccc|ccc} g_1(u_1) & \cdots & g_1(u_n) & f_1(w_{m-n}) & \cdots & f_1(w_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ g_m(u_1) & \cdots & g_m(u_n) & f_m(w_{m-n}) & \cdots & f_m(w_1) \end{array} \right)$$

- Assuming $\{v_1, \ldots, v_m\}$ are Bethe roots, we have

$$S(\{u\}_n | \{v\}_m) = \frac{\prod_{i=1}^{m} \prod_{j=1}^{\ell} (v_i - w_j + 1)}{\prod_{i=1}^{n} \prod_{j=1}^{m-n} (u_i - w_j + 1)} \frac{\det \mathcal{M}}{\Delta\{u\}_n \Delta\{v\}_m \Delta\{w\}_{m-n}}$$

Properties of $sl(3)$ XXX scalar product

- We define the first set of intermediate scalar products:

$$S(\{\nu\}_k|\{v\}_m,\{u\}_n) =$$

$$\begin{cases} \langle 1^{\ell-m+k}, 2^{m-n-k}, 3^n | t_{21}(\nu_k)\dots t_{21}(\nu_1)|\{v\}_m,\{u\}_n\rangle \\ \langle 1^{\ell-m+k}, 3^{m-k} | t_{31}(\nu_k)\dots t_{31}(\nu_{m-n+1}) t_{21}(\nu_{m-n})\dots t_{21}(\nu_1)|\{v\}_m,\{u\}_n\rangle \end{cases}$$

applying to the cases $k \leq m-n$ and $k > m-n$.

- The second set of intermediate scalar products:

$$S(\{\mu\}_k,\{\nu\}_m|\{v\}_m,\{u\}_n) = \langle 3^{n-k}, 2^{m-n+k}|_\alpha \otimes \langle 1^\ell|$$

$$\times\, C^{(2)}(\{\nu\}_m|\mu_k)\dots C^{(2)}(\{\nu\}_m|\mu_1) C^{(1)}_{\alpha_m}(\nu_m)\dots C^{(1)}_{\alpha_1}(\nu_1)|\{v\}_m,\{u\}_n\rangle$$

Properties of $sl(3)$ XXX scalar product

- Again, we have a set of uniquely-determining properties. $S(\{\nu\}_k|\{v\}_m,\{u\}_n)$
  1. Is a polynomial in ν_k of degree $\ell-1$, with zeros at $\nu_k = w_i$ for all $1 \leq i \leq m-k$.
  2. Is symmetric in the variables $\{w_{m-k+1},\ldots,w_\ell\}$.
  3. Satisfies the recursion relation

$$S(\{\nu\}_k|\{v\}_m,\{u\}_n)\Big|_{\nu_{k+1}=w_{m-k+1}} = -\prod_{i=1}^{\ell}(w_{m-k+1}-w_i-1)S(\{\nu\}_{k-1}|\{v\}_m,\{u\}_n)$$

  4.

$$S(\{\nu\}_0|\{v\}_m,\{u\}_n) = \prod_{i=1}^{m}\prod_{j=m+1}^{\ell}(v_i-w_j+1)\prod_{i=1}^{n}\prod_{j=1}^{\ell}(u_i-w_j)X(\{u\}_n,\{v\}_m|\{w\}_m)$$

Properties of $sl(3)$ XXX scalar product

- Also, we find that $S(\{\mu\}_k, \{\nu\}_m | \{v\}_m, \{u\}_n)$
  1. Is a polynomial in μ_k of degree $\ell + m - 1$, with zeros at

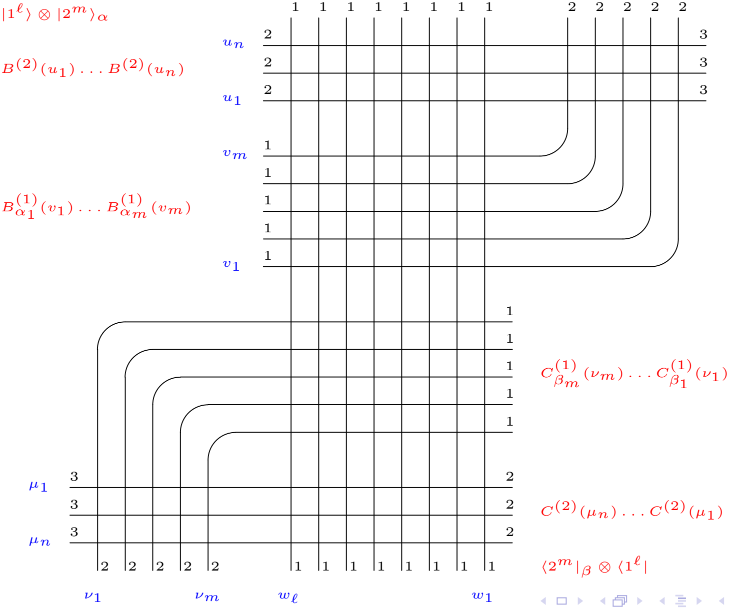
$$\mu_k = w_i, \text{ for all } 1 \leq i \leq \ell$$
$$\mu_k = \nu_{m-n+k+j}, \text{for all } 1 \leq j \leq n-k$$

  2. Is symmetric in the variables $\{\nu_1, \ldots, \nu_{m-n+k}\}$.
  3. Satisfies the recursion relation

$$S(\{\mu\}_k, \{\nu\}_m | \{v\}_m, \{u\}_n)\Big|_{\mu_k+1=\nu_{m-n+k+1}} =$$
$$-\prod_{i=1}^{m} (\nu_{m-n+k+1} - \nu_i - 1) \prod_{j=1}^{\ell} (\nu_{m-n+k+1} - w_j - 1) S(\{\mu\}_{k-1}, \{\nu\}_m | \{v\}_m, \{u\}_n)$$

  4. $S(\{\mu\}_0, \{\nu\}_m | \{v\}_m, \{u\}_n) = S(\{\nu\}_m | \{v\}_m, \{u\}_n)$.

Recursion procedure – $sl(3)$ scalar product



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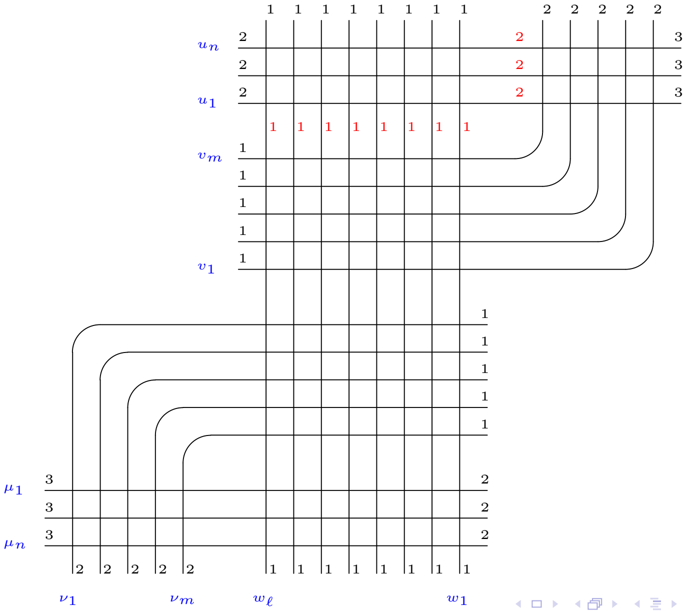

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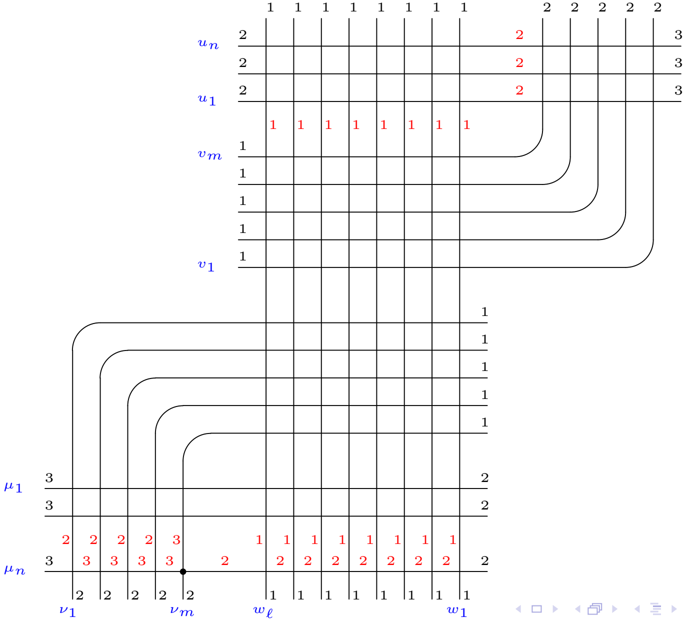

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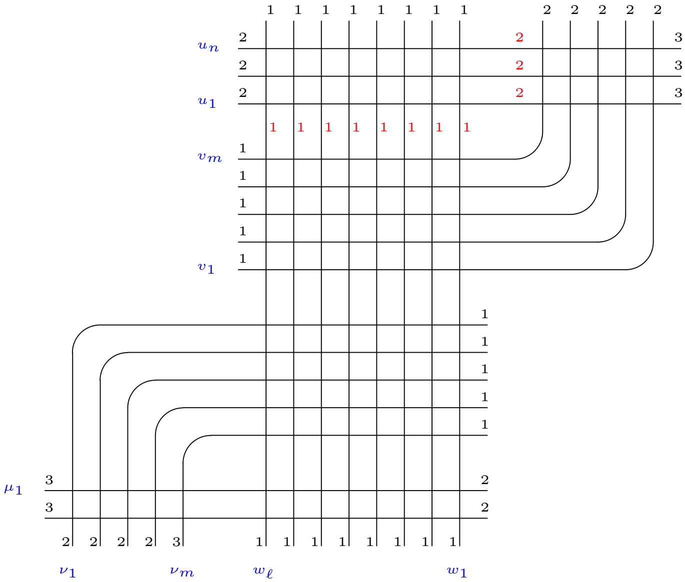

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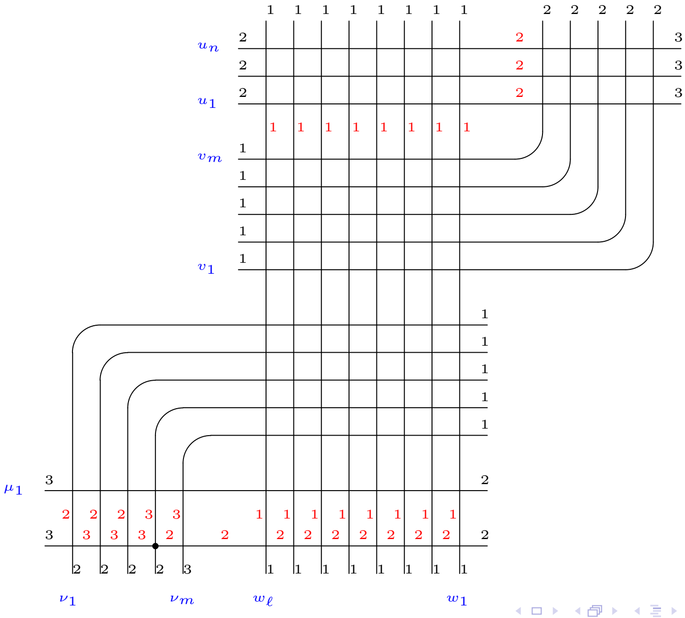

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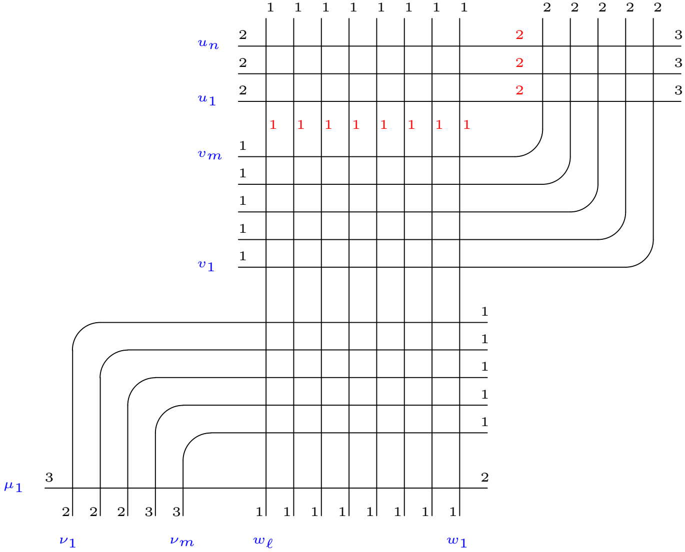

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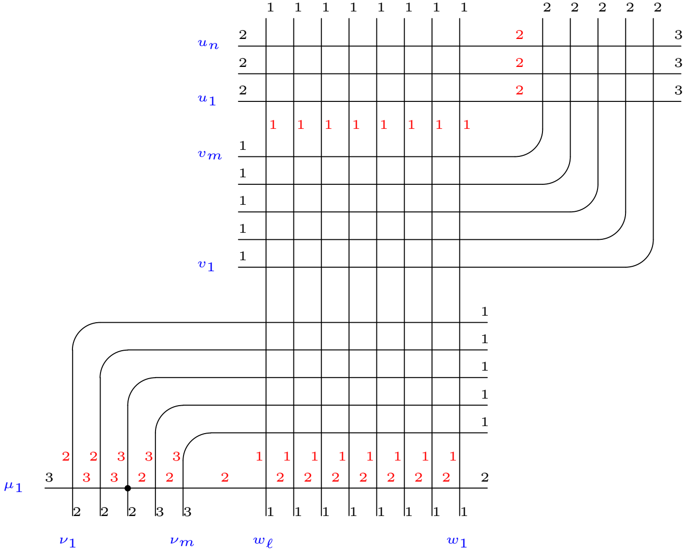

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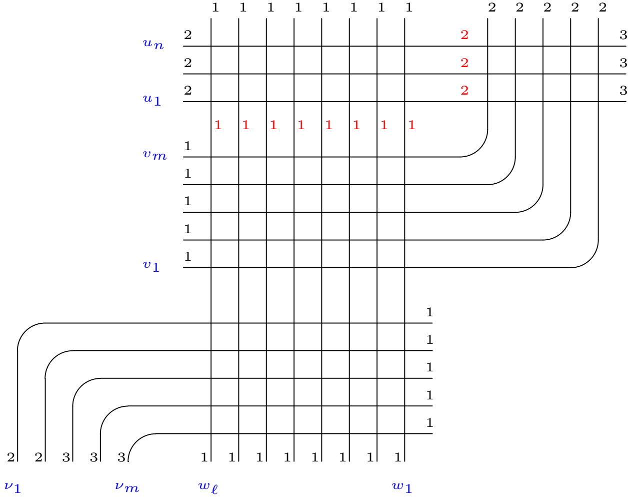

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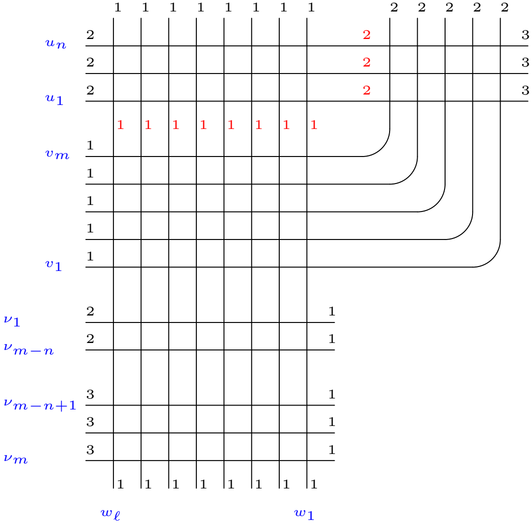

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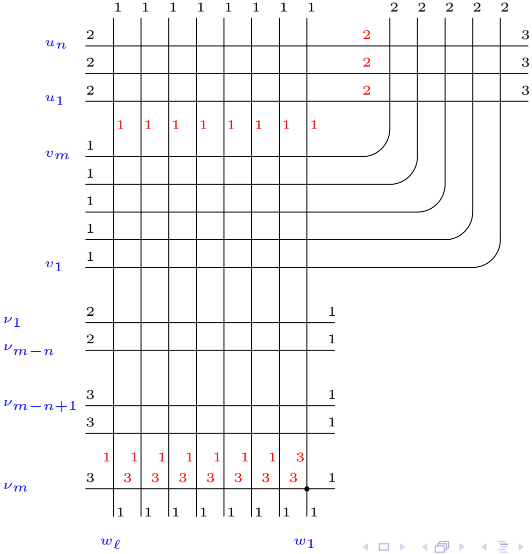

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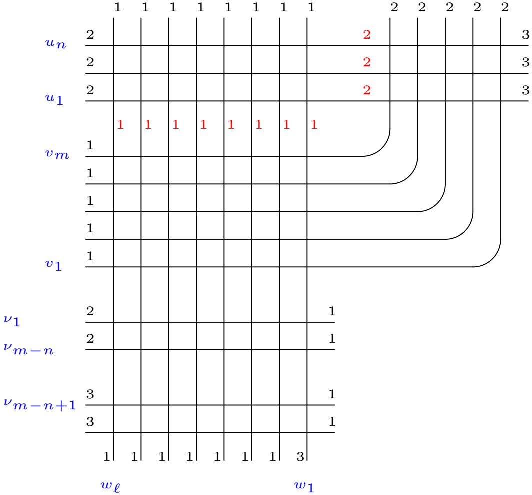

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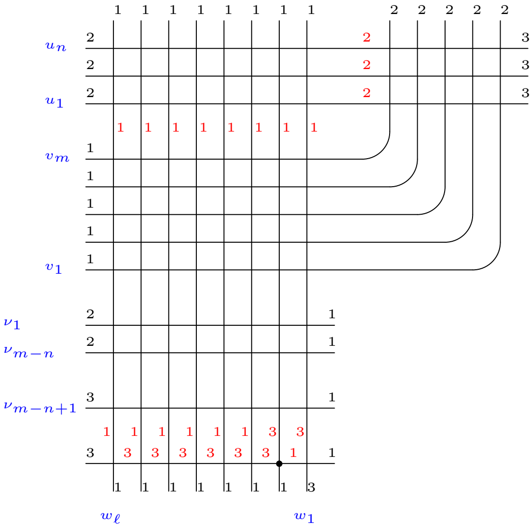

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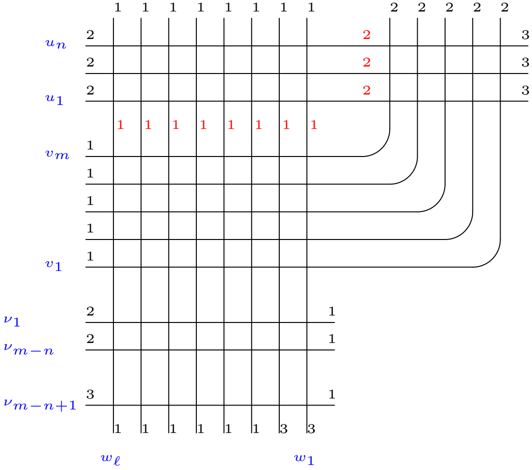

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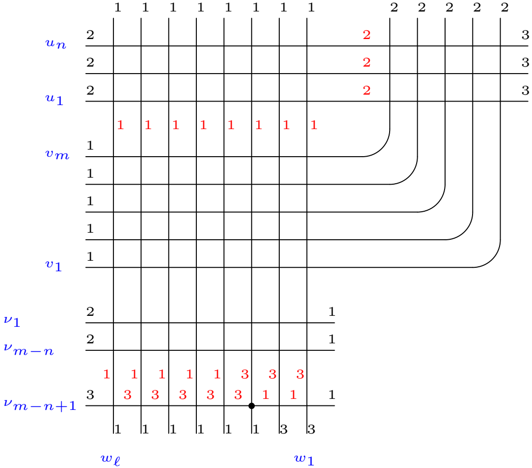
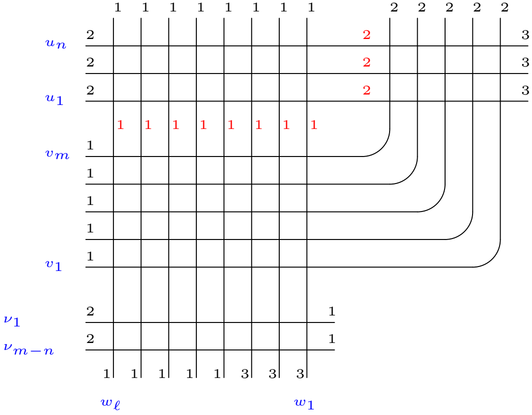
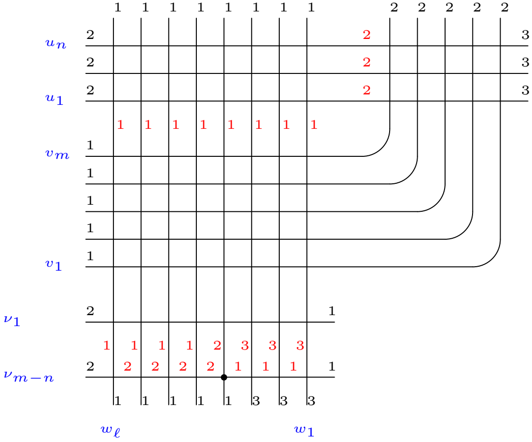


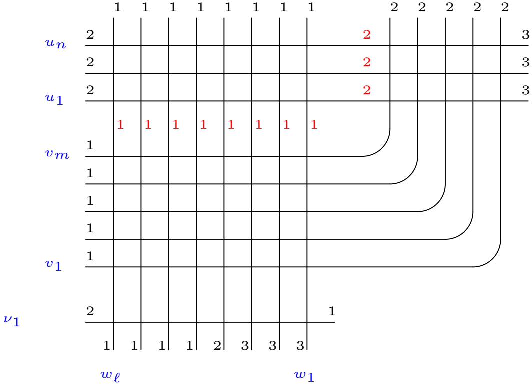
Recursion procedure – $sl(3)$ scalar product



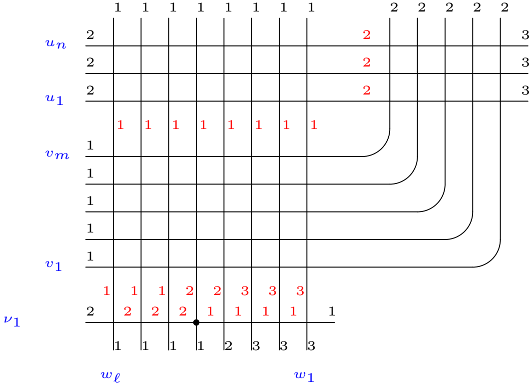
Recursion procedure – $sl(3)$ scalar product



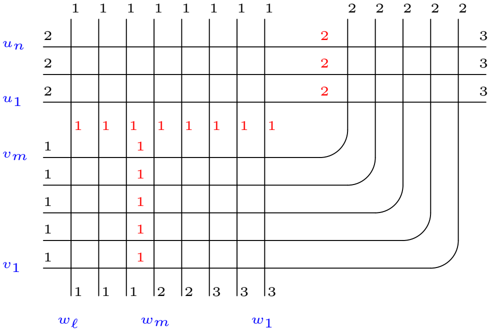
Recursion procedure – $sl(3)$ scalar product



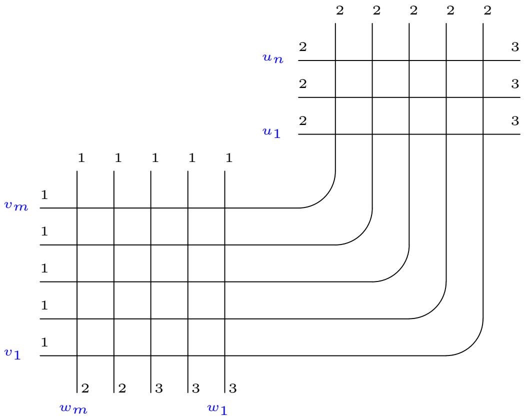
Recursion procedure – $sl(3)$ scalar product



End of recursion – $sl(3)$ scalar product



$sl(3)$ analogue of domain wall partition function



$sl(3)$ analogue of domain wall partition function

- The quantity $X(\{u\}_n, \{v\}_m | \{w\}_m)$ can be expressed as a multiple integral

$$X\left(\{u\}_n, \{v\}_m \middle| \{w\}_m\right) = \frac{1}{n!(m-n)!} \oint_{\mathcal{C}_1} \frac{dz_1}{2\pi\sqrt{-1}} \cdots \oint_{\mathcal{C}_m} \frac{dz_m}{2\pi\sqrt{-1}}$$

$$\frac{\prod_{i \neq j}(z_i - z_j)}{\prod_{i,j=1}^{m}(z_i - v_j)} Z\left(\{u\}_n^1 \middle| \{z\}_n^1\right) Z\left(\{z\}_n^1 \middle| \{w\}_n^1\right) Z\left(\{z\}_m^{n+1} \middle| \{w\}_m^{n+1}\right)$$

$$\prod_{i=1}^{n} \prod_{j=n+1}^{m} (u_i - z_j + 1)(z_i - w_j + 1)(z_j - w_i) \frac{(z_j - z_i + 1)}{(z_j - z_i)}$$

where $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are concentric, enclosing the poles at $\{v_1, \ldots, v_m\}$.