Long-distance asymptotic behavior of temperature correlation functions in the one-dimensional Bose gas

K. K. Kozlowski, J. M. Maillet, N. A. Slavnov

[1] *Long-distance behavior of temperature correlation functions in the one-dimensional Bose gas*, J. Stat. Mech. (2011) P03018.

[2] *Correlation functions of one-dimensional bosons at low temperature*, J. Stat. Mech. (2011) P03019.

Quantum Nonlinear Schrödinger equation

$$[\phi(x,t), \phi^\dagger(y,t)] = \delta(x-y)$$

$$\langle 0|\phi^\dagger(x) = 0, \qquad \phi(x)|0\rangle = 0$$

$$H = \int_0^L \left(\partial_x\phi^\dagger\partial_x\phi + c\phi^\dagger\phi^\dagger\phi\phi - h\phi^\dagger\phi\right) dx$$

$c > 0$ is a coupling constant

$h > 0$ is a chemical potential

Eigenfunctions

$$H|\psi\rangle = E|\psi\rangle$$

$$|\psi\rangle = |\psi(\lambda_1, \ldots, \lambda_N)\rangle, \qquad N = 0, 1, \ldots$$

The eigenfunctions $|\psi\rangle$ and their corresponding eigenvalues E can be found by Bethe Ansatz.

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Temperature correlation functions

$$\langle \mathcal{O} \rangle_T = \frac{\mathrm{tr}\left(\mathcal{O} e^{-H/T}\right)}{\mathrm{tr}\left(e^{-H/T}\right)} = \frac{\sum \langle \psi | \mathcal{O} | \psi \rangle e^{-E/T}}{\sum e^{-E/T}}$$

Generating function for density-density correlations

The object of study:

$$\langle j(x) j(0) \rangle_T, \qquad \text{with} \qquad j(x) = \phi^\dagger(x)\, \phi(x)$$

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$$\langle e^{2\pi i \alpha \mathcal{Q}_x} \rangle_T = \frac{\sum \langle \psi | e^{2\pi i \alpha \mathcal{Q}_x} | \psi \rangle e^{-E/T}}{\sum e^{-E/T}}$$

$$\langle j(x) j(0) \rangle_T = \frac{-1}{8\pi^2} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial \alpha^2} \langle e^{2\pi i \alpha \mathcal{Q}_x} \rangle_T \Big|_{\alpha=0}$$

Asymptotic behavior at $x \to \infty$

$$\langle j(x)j(0)\rangle_T = D^2 + \sum B_i e^{-xp_i}, \qquad x \to \infty$$

Here D is the average density of the gas. One should find the correlation lengths p_i and the amplitudes B_i.

Asymptotic behavior at $x \to \infty$

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Multiple integrals approach

N.M. Bogolyubov, A.G. Izergin, V.E. Korepin, 1985

$$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T = \frac{\sum \langle \psi| e^{2\pi i\alpha \mathcal{Q}_x}|\psi\rangle e^{-E/T}}{\sum e^{-E/T}} = \langle \psi_T| e^{2\pi i\alpha \mathcal{Q}_x}|\psi_T\rangle$$

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$$\langle j(x)j(0)\rangle_T = D^2 + \sum_{n=2}^{\infty} \Gamma_n$$

$$\Gamma_n = \int d\lambda_1 \cdots d\lambda_n \, G_n(x|\lambda_1, \ldots, \lambda_n)$$

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Multiple integrals approach

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Large x asymptotic analysis of Γ_2, Γ_3 and Γ_4 led the authors (and N.S.) to a wrong conjecture on the leading p_i.

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Dual fields approach

V.E. Korepin, 1988

$$\langle e^{2\pi i\alpha\mathcal{Q}_x}\rangle_T = (0|\,\mathrm{det}\Big(I + V(\Phi_i)\Big)|0), \quad \Phi_i \quad \text{are dual fields}$$

$$[\Phi_j, \Phi_k] = 0, \qquad (0|\prod \Phi_i|0) \neq 0$$

Asymptotic behavior at $x \to \infty$

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The expectation value of the asymptotic behavior is not necessary equal to the asymptotic behavior of the expectation value.

Asymptotic behavior at $x \to \infty$

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Quantum Transfer Matrix approach

A. Seel, T. Bhattacharyya, F. Göhmann, A. Klümper, 2007

Asymptotic behavior at $x \to \infty$

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Quantum Transfer Matrix approach

A. Seel, T. Bhattacharyya, F. Göhmann, A. Klümper, 2007

The quantum Bose-gas is considered as a special limit of the XXZ spin-1/2 chain. The correlation lengths are related to the eigenvalues of the Quantum Transfer matrix: $\log(\Lambda_i/\Lambda_{max}) \to p_i$. The amplitudes B_i are given as form factors of $j(x)j(0)$ with respect to the Quantum Transfer matrix eigenfunctions.

Asymptotic behavior at $x \to \infty$

$$\langle j(x)j(0)\rangle_T = D^2 + \sum B_i e^{-xp_i}, \qquad x \to \infty$$

Multiple integrals approach from master equation

N. Kitanine, K. K. Kozlowski, J. M. Maillet, N.S., V. Terras, 2009

$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_0 = \langle \psi_0 | e^{2\pi i\alpha \mathcal{Q}_x} | \psi_0\rangle$ — zero temperature

K. K. Kozlowski, J. M. Maillet, N.S., 2011

$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T = \langle \psi_T | e^{2\pi i\alpha \mathcal{Q}_x} | \psi_T\rangle$ — finite temperature

Master equation

$$\langle\psi|e^{2\pi i\alpha\mathcal{Q}_x}|\psi\rangle = \oint \mathcal{G}(x|z_1,\ldots,z_N,\lambda_1,\ldots,\lambda_N)dz_1\cdots dz_N$$

$$|\psi\rangle = |\psi(\lambda_1,\ldots,\lambda_N)\rangle$$

Master equation

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Unsolved problem: to take the thermodynamic limit

$L\to\infty$, $N\to\infty$, $N/L=D$

directly in the master equation. Instead, one can use different expansions of the master equation into series.

Master equation

$$\langle\psi|e^{2\pi i\alpha\mathcal{Q}_x}|\psi\rangle = \oint \mathcal{G}(x|z_1,\dots,z_N,\lambda_1,\dots,\lambda_N)dz_1\cdots dz_N$$

Correlation function in the thermodynamic limit

$$\langle e^{2\pi i\alpha\mathcal{Q}_x}\rangle = \sum_{n=0}^{\infty} U_n(x)$$

$$U_n(x) = \oint dz_1\cdots dz_n \int d\lambda_1\cdots d\lambda_n\, G_n(x|z_1,\dots,z_n,\lambda_1,\dots,\lambda_n)$$

Zero temperature **Finite temperature**

$$\int_{-q}^{q} d\lambda \qquad \longrightarrow \qquad \int_{\mathbb{R}} \vartheta(\lambda)\, d\lambda$$

Fermi weight: $\vartheta(\lambda) = \left(1 + e^{\varepsilon(\lambda)/T}\right)^{-1}$

Dressed energy $\varepsilon(\lambda)$:

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi} \int_{\mathbb{R}} K(\lambda - \mu) \log\left(1 + e^{-\varepsilon(\mu)/T}\right) d\mu$$

$$K(\lambda) = \frac{2c}{c^2 + \lambda^2}$$

Zero temperature **Finite temperature**

$$\int_{-q}^{q} d\lambda \qquad \longleftrightarrow \qquad \int_{\mathbb{R}} \vartheta(\lambda)\, d\lambda$$

Fermi weight: $\vartheta(\lambda) = \left(1 + e^{\varepsilon(\lambda)/T}\right)^{-1}$

$$\vartheta(\lambda)\Big|_{T=0} = \chi_{[-q,q]}(\lambda), \qquad \varepsilon(\pm q) = 0$$

Zero temperature **Finite temperature**

$$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_0 = \mathcal{N}_0^{-1}\sum_{n=0}^{\infty} U_n^{(0)}(x) \qquad \longleftrightarrow \qquad \langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T = \mathcal{N}_T^{-1}\sum_{n=0}^{\infty} U_n^{(T)}(x)$$

$$U_n^{(0)}(x) = \oint \prod_{j=1}^{n} dz_j \int_{-q}^{q} \prod_{j=1}^{n} d\lambda_j \cdot G_n(x|z_1,\ldots,z_n,\lambda_1,\ldots,\lambda_n)$$

$$U_n^{(T)}(x) = \oint \prod_{j=1}^{n} dz_j \int_{\mathbb{R}} \prod_{j=1}^{n} \vartheta(\lambda_j) d\lambda_j \cdot G_n(x|z_1,\ldots,z_n,\lambda_1,\ldots,\lambda_n)$$

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Unsolved problem

The convergence of the multiple integrals series is not proved because of complicated structure of the integrands G_n. Therefore all below transforms of this series are rather formal.

$$\langle e^{2\pi i\alpha Q_x}\rangle_T = \mathcal{N}_T^{-1}\sum_{n=0}^{\infty} U_n^{(T)}(x)$$

$$U_n^{(T)}(x) = \oint \prod_{j=1}^{n} dz_j \int_{\mathbb{R}} \prod_{j=1}^{n} \vartheta(\lambda_j)d\lambda_j \cdot G_n(x|z_1,\ldots,z_n,\lambda_1,\ldots,\lambda_n)$$

Unsolved problem

The convergence of the multiple integrals series is not proved because of complicated structure of the integrands G_n. Therefore all below transforms of this series are rather formal

The arguments for the convergence

1. In the free fermion point $c=\infty$ the series turns into the expansion for a Fredholm determinant.
2. Reasonable results.

.

$$U_n^{(T)}(x) = \oint \prod_{j=1}^{n} dz_j \int_{\mathbb{R}} \prod_{j=1}^{n} \vartheta(\lambda_j) d\lambda_j \ \cdot G_n(x|z_1, \ldots, z_n, \lambda_1, \ldots, \lambda_n)$$

The multiple integrals above can be estimated at x large

$$U_n^{(T)}(x) \stackrel{x\to\infty}{\longrightarrow} \mathcal{U}_n^{(T)}(x) + \textit{corrections}(x,n)$$

$$\langle e^{2\pi i \alpha \mathcal{Q}_x} \rangle_T \stackrel{x\to\infty}{\longrightarrow} \mathcal{N}_T^{-1} \sum_{n=0}^{\infty} \mathcal{U}_n^{(T)}(x)$$

provided the *corrections*(x,n) are uniform over n

Generating functional for the multiple integrals $U_n^{(T)}(x)$

$$\partial_\gamma^n \, \mathrm{det}(I + \gamma V)\Big|_{\gamma=0} \Rightarrow U_n^{(T)}(x)$$

Generating functional for the multiple integrals $U_n^{(T)}(x)$

$$\partial_\gamma^n \det(I+\gamma V)\Big|_{\gamma=0} \Rightarrow U_n^{(T)}(x)$$

Generalized Sine-kernel

$$V(\lambda,\mu) = \frac{e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)}{2\pi i(\lambda-\mu)} F(\lambda)\vartheta(\lambda)$$

$$e_\pm(\lambda) = e^{\pm ixp(\lambda)\pm g(\lambda)}$$

Functions $p(\lambda)$, $g(\lambda)$ and $F(\lambda)$ supposed to be holomorphic in a neighborhood of the real axis, say, $|\Im(\lambda)| < a$. The Fermi weight $\vartheta(\lambda)$ has simple poles $\vartheta^{-1}(r_j^\pm) = 0$ within the same strip.

Large x asymptotics $\mathcal{U}_n^{(T)}(x)$ of the multiple integrals $U_n^{(T)}(x)$ follows from the large x asymptotic behavior of the Fredholm determinant $\det(I+\gamma V)$:

$$\partial_\gamma^n \det(I+\gamma V)\Big|_{\gamma=0} \overset{x\to\infty}{\Rightarrow} \mathcal{U}_n^{(T)}(x)$$

At zero temperature the integral $I+\gamma V$ acts on the finite interval $[-q,q]$. The most complicated part of the asymptotics comes from the ends of the interval $\pm q$. At finite temperature the integral $I+\gamma V$ acts on the real axis $\mathbb{R}$. Due to this property the asymptotic behavior of $\det(I+\gamma V)$ has much more simple form

$$\det(I+\gamma V) \overset{x\to\infty}{=} \sum_i h_i e^{-x\alpha_i}\Big(1+O(e^{-ax})\Big)$$

Generalized Sine-kernel

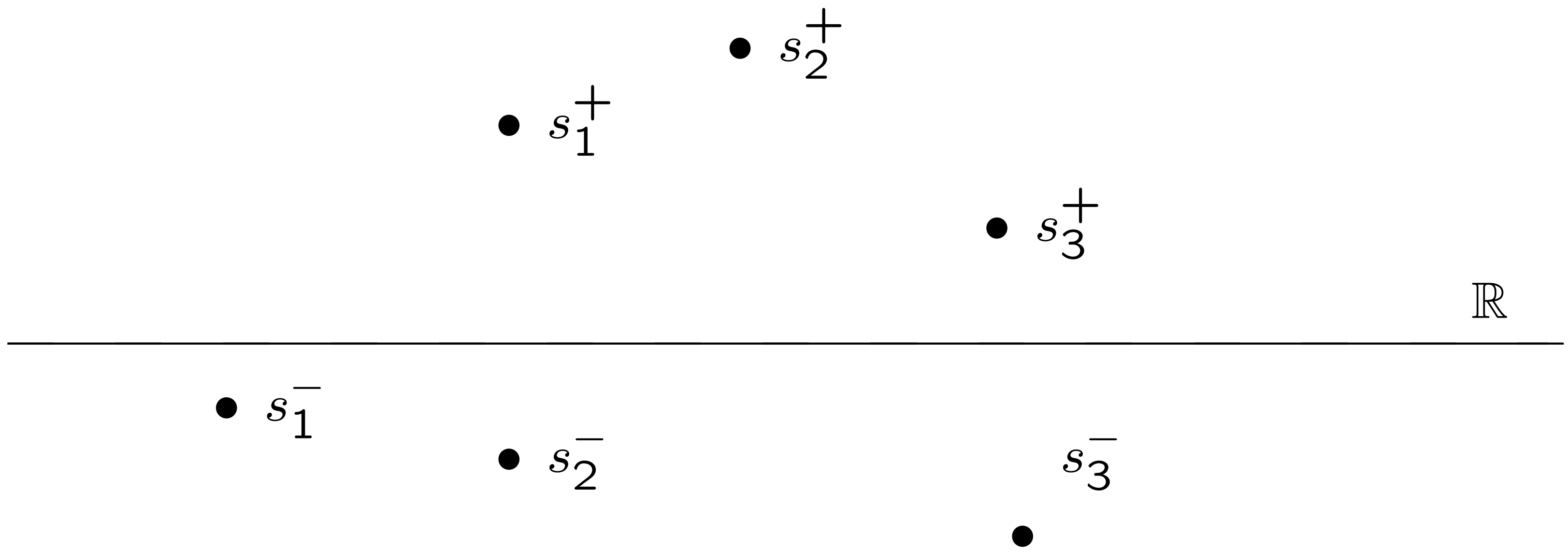
$$V(\lambda, \mu) = \frac{e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)}{2\pi i(\lambda - \mu)} F(\lambda)\vartheta(\lambda)$$

$$e_\pm(\lambda) = e^{\pm ixp(\lambda) \pm g(\lambda)}$$

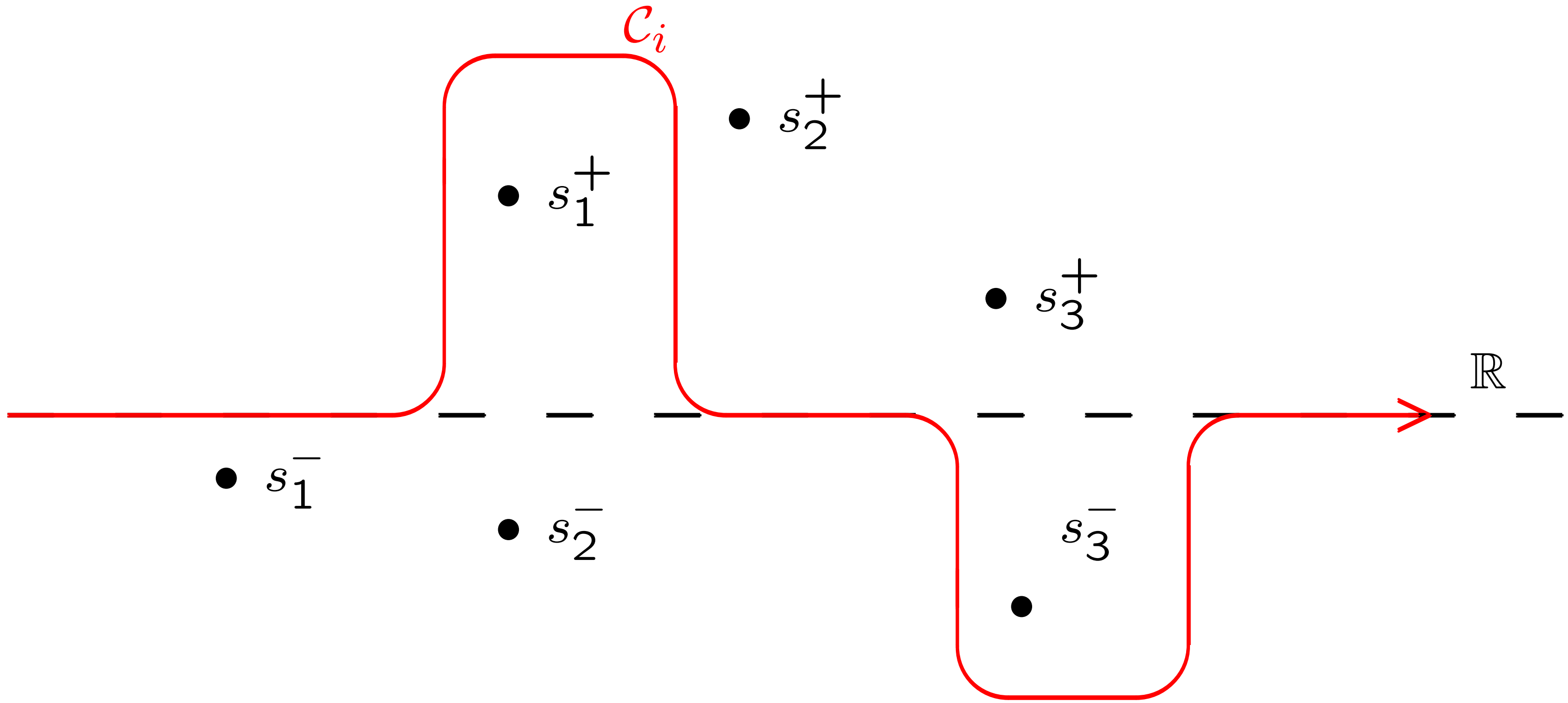
Define:

$$\nu(\lambda) = \frac{-1}{2\pi i} \log\Big(1 + \gamma\vartheta(\lambda)F(\lambda)\Big)$$

$$\mathcal{A}_{\mathcal{L}}([g],[\nu]) = -\int_{\mathcal{L}} \Big(ix + g'(\lambda)\Big)\nu(\lambda)\, d\lambda + \iint_{\mathcal{L}} \frac{\nu(\lambda)\nu(\mu)}{(\lambda - \mu_+)^2}\, d\lambda\, d\mu$$



$$1 + \gamma\vartheta(s_j^{\pm})F(s_j^{\pm}) = 0$$



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$$\mathcal{A}_{\mathcal{C}_i}([g],[\nu]) = -\int_{\mathcal{C}_i} \Big(ix + g'(\lambda)\Big)\nu(\lambda)\,d\lambda + \iint_{\mathcal{C}_i} \frac{\nu(\lambda)\nu(\mu)}{(\lambda-\mu_+)^2}\,d\lambda\,d\mu$$

$$\det(I+\gamma V) = \sum_i \exp\Big(\mathcal{A}_{\mathcal{C}_i}([g],[\nu])\Big) \cdot \Big[1 + O\left(e^{-ax}\right)\Big]$$

Asymptotic behavior of the generating function

$$\langle e^{2\pi i\alpha\mathcal{Q}_x}\rangle_T \quad = \quad \mathcal{N}_T^{-1}\sum_{n=0}^{\infty} U_n^{(T)}(x)$$

Asymptotic behavior of the generating function

$$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T \quad \xrightarrow{x\to\infty} \quad \mathcal{N}_T^{-1} \sum_{n=0}^{\infty} \mathcal{U}_n^{(T)}(x)$$

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The series of $\mathcal{U}_n^{(T)}(x)$ is the **Generalized Lagrange series**. The result of its summation can be expressed in terms of the unique solution of certain integral equation (provided the series is convergent).

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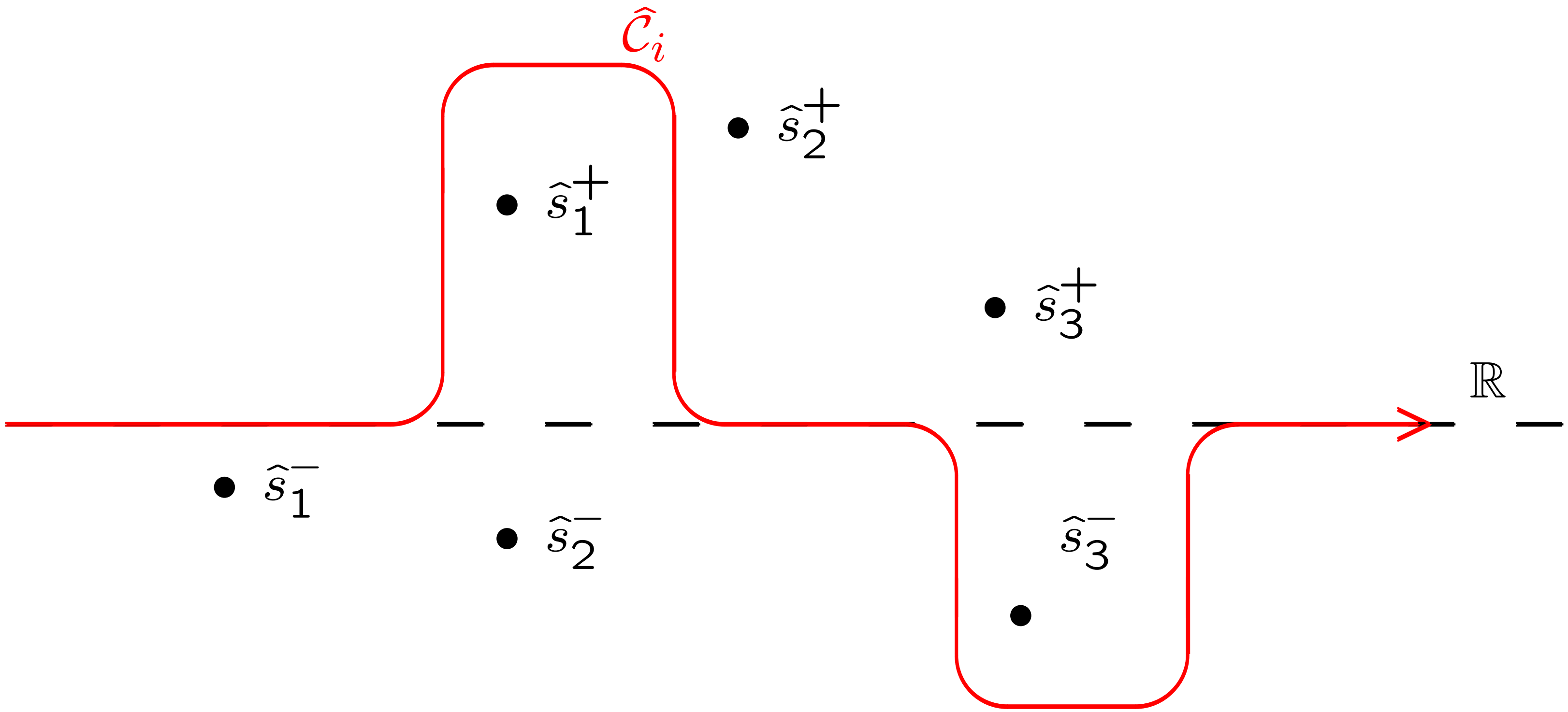
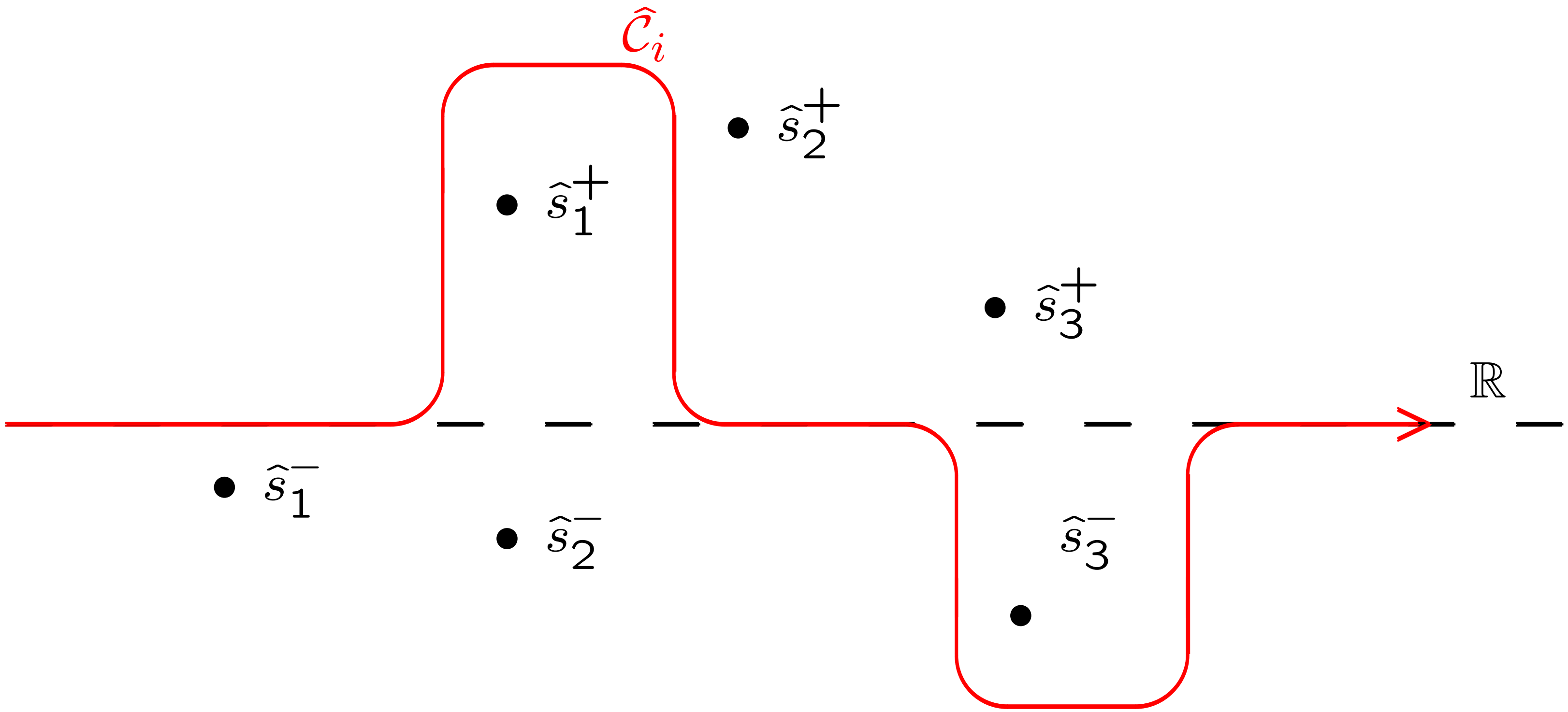
The series of $\mathcal{U}_n^{(T)}(x)$ is the **Generalized Lagrange series**. The result of its summation can be expressed in terms of the unique solution of certain integral equation (provided the series is convergent).

$$\sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{d^n}{d\omega^n}\Big(f^n(\omega)F(\omega)\Big)\Big|_{\omega=0} = \frac{F(z)}{1-f'(z)}, \qquad\qquad z=f(z)$$



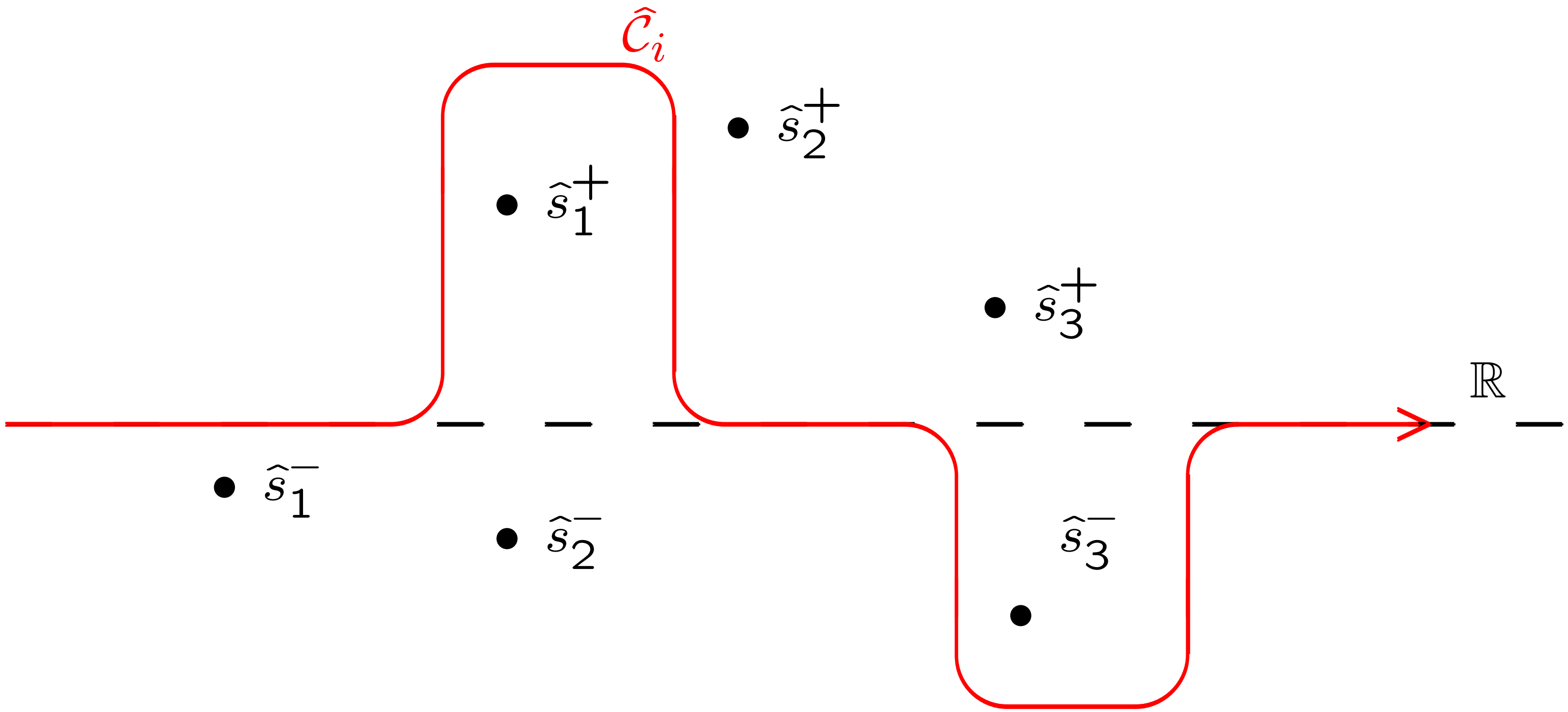
$$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T \longrightarrow \sum_{\widehat{\mathcal{C}}_i} e^{-xp_i} B[u_i], \qquad x \to \infty$$

$$1 + e^{-u_i(\widehat{s}_\ell^{\pm})/T} = 0$$



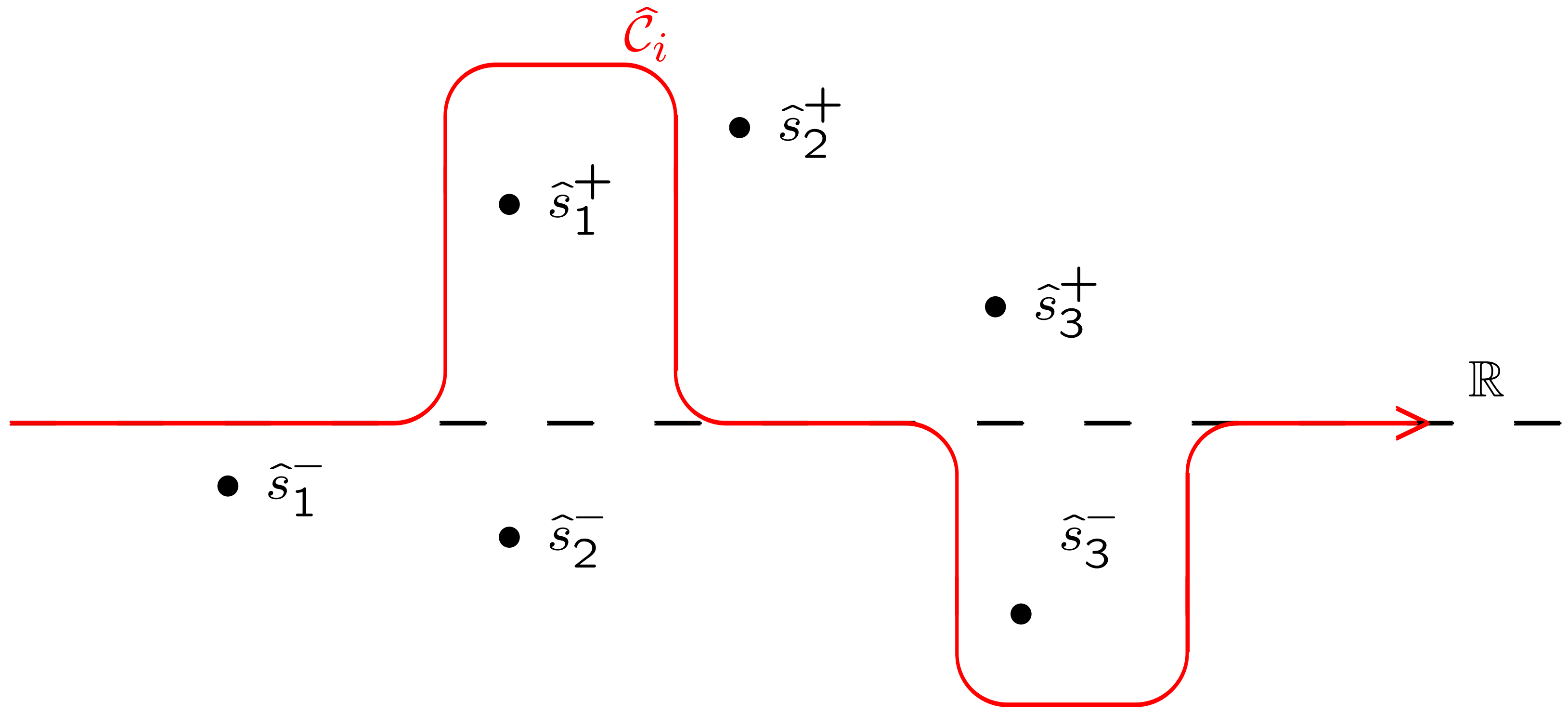
$$u_i(\lambda) = \lambda^2 - (h + 2\pi i \alpha T) - \frac{T}{2\pi} \int_{\widehat{\mathcal{C}}_i} K(\lambda - \mu) \log\left(1 + e^{-\frac{u_i(\mu)}{T}}\right) d\mu$$

$$1 + e^{-u_i(\widehat{s}_\ell^{\pm})/T} = 0, \qquad e^{-u_i(\lambda)/T} \longrightarrow \mathfrak{a}_i(\lambda)$$



$$\langle e^{2\pi i \alpha \mathcal{Q}_x} \rangle_T \longrightarrow \sum_{\hat{\mathcal{C}}_i} e^{-x p_i} B[u_i], \qquad x \to \infty$$

$$p_i = -\frac{1}{2\pi} \int_{\hat{\mathcal{C}}_i} \log \left(\frac{1 + e^{-\frac{u_i(\lambda)}{T}}}{1 + e^{-\frac{\varepsilon(\lambda)}{T}}} \right) d\lambda, \qquad p_i \longrightarrow \log \frac{\Lambda_i(h + 2\pi i \alpha T)}{\Lambda_0(h)}$$



$$\langle e^{2\pi i \alpha \mathcal{Q}_x} \rangle_T \longrightarrow \sum_{\widehat{\mathcal{C}}_i} e^{-xp_i} B[u_i], \qquad x \to \infty$$

The amplitudes $B[u_i]$ are rather cumbersome functionals of $u_i(\lambda)$ (and associated contours $\widehat{\mathcal{C}}_i$). Most probably they are related to the form factors of the operator $e^{2\pi i \alpha \mathcal{Q}_x}$ between the thermal state and some excited state.

Asymptotics of the
Fredholm determinant
(free fermions)

$$\Downarrow$$

Dressing via the
Generalized Lagrange
series

$$\Downarrow$$

Asymptotics of the
correlation function

Low-temperature limit

$$\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi} \int_{\mathbb{R}} K(\lambda - \mu) \log \left(1 + e^{-\frac{\varepsilon(\mu)}{T}}\right) d\mu$$

$$\varepsilon(\lambda) = \sum_{k \geq 0} T^k \varepsilon_k(\lambda)$$

Low-temperature limit

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$$\varepsilon(\lambda) = \sum_{k \geq 0} T^k \varepsilon_k(\lambda)$$

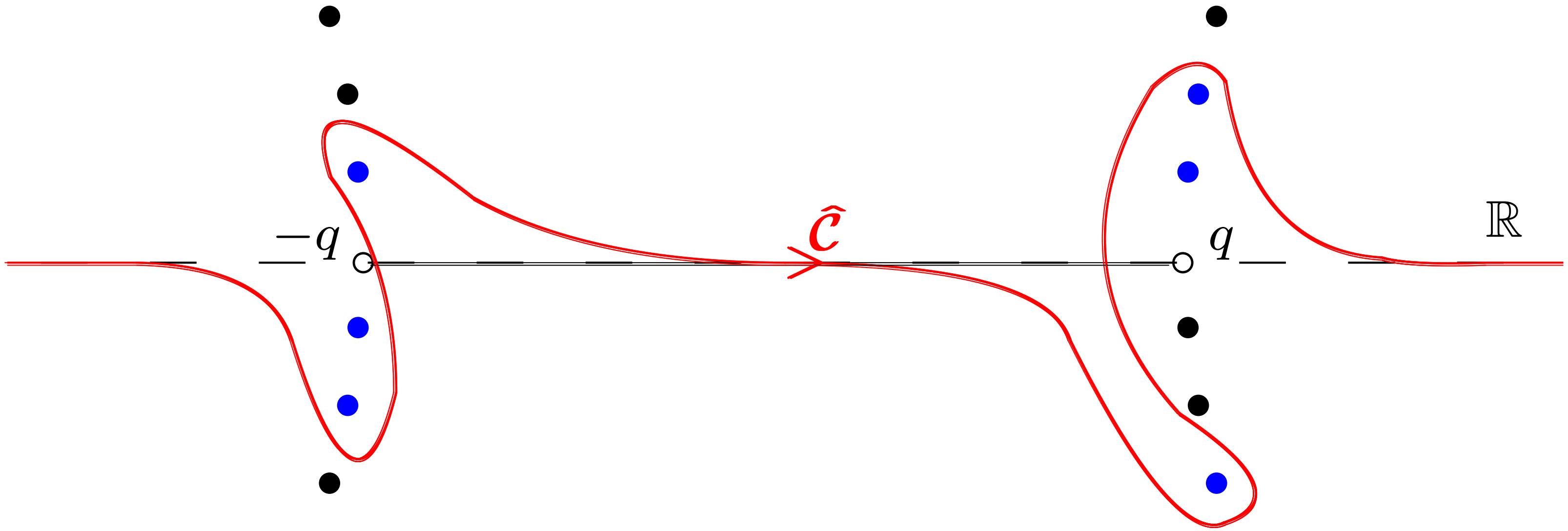
$$\varepsilon_0(\lambda) = \lambda^2 - h + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu) \varepsilon_0(\mu)\, d\mu, \qquad \varepsilon_0(\pm q) = 0$$

$$\varepsilon_1(\lambda) = 0$$

$$\varepsilon_2(\lambda) = -\frac{\pi^2}{6\varepsilon_0'} \Big(R(\lambda, q) + R(\lambda, -q) \Big), \qquad \varepsilon_0' = \varepsilon_0'(q)$$

$R(\lambda, \mu)$ is the resolvent of the operator $I - \frac{1}{2\pi} K$

Low-temperature limit

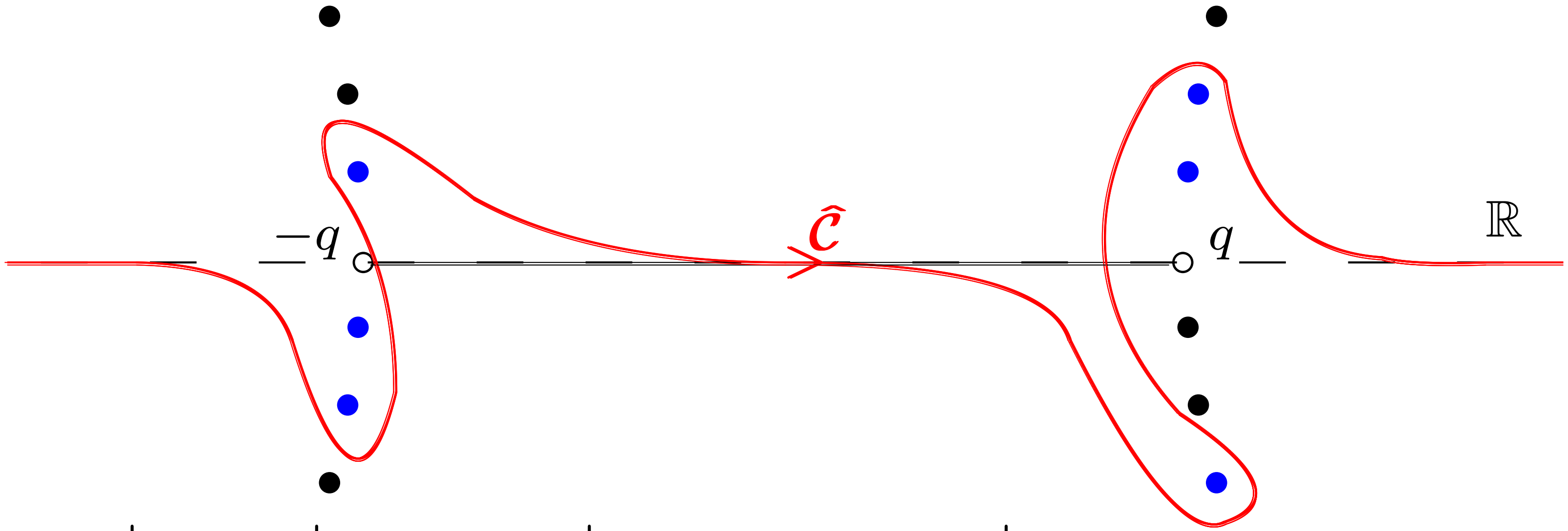


$$u(\lambda) = \lambda^2 - (h + 2\pi i \alpha T) - \frac{T}{2\pi} \int_{\hat{\mathcal{C}}} K(\lambda - \mu) \log\left(1 + e^{-\frac{u(\mu)}{T}}\right) d\mu$$

$$u(\lambda) = \sum_{k \geq 0} T^k u_k(\lambda)$$

$$\hat{s}^{\pm} = -q + \sum_{k \geq 1} T^k r_k \qquad \text{or} \qquad \hat{s}^{\pm} = q + \sum_{k \geq 1} T^k \tilde{r}_k$$

Low-temperature limit



$$\#\{\hat{s}^+\} = n_p^+, \quad if \quad \hat{s}^+ \to +q, \qquad (n_p^+ = 2)$$

$$\#\{\hat{s}^+\} = n_p^-, \quad if \quad \hat{s}^+ \to -q, \qquad (n_p^- = 1)$$

$$\#\{\hat{s}^-\} = n_h^+, \quad if \quad \hat{s}^- \to +q, \qquad (n_h^+ = 1)$$

$$\#\{\hat{s}^-\} = n_h^-, \quad if \quad \hat{s}^- \to -q, \qquad (n_h^- = 2)$$

$$n_p^+ - n_h^+ = n_h^- - n_p^- = \ell, \qquad (\ell = 1)$$

$$n_p^+ + n_p^- = n_h^+ + n_h^- = n, \qquad (n = 3)$$

Solutions

$$u_0(\lambda) = \varepsilon_0(\lambda), \qquad u_1(\lambda) = -2\pi i \alpha_\ell Z(\lambda) + 2\pi i \ell$$

Here $\alpha_\ell = \alpha + \ell$ and $Z(\lambda)$ is the dressed charge:

$$Z(\lambda) = 1 + \frac{1}{2\pi} \int_{-q}^{q} K(\lambda - \mu) Z(\mu)\, d\mu$$

$$\hat{s}^+ = \pm q + \frac{iT}{\varepsilon_0'} \Big(2\pi (p^\pm - 1/2) \pm i u_1(q) \Big) + O(T^2)$$

$$\hat{s}^- = \pm q - \frac{iT}{\varepsilon_0'} \Big(2\pi (h^\pm - 1/2) \mp i u_1(q) \Big) + O(T^2)$$

Here $p^\pm$ and $h^\pm$ are integers.

Correlation lengths

$$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T \longrightarrow \sum_i e^{-xp[u_i]}B[u_i], \qquad x\to\infty$$

$$p[u] = -\frac{1}{2\pi}\int_{\hat{\mathcal{C}}} \log\left(\frac{1+e^{-\frac{u(\lambda)}{T}}}{1+e^{-\frac{\varepsilon(\lambda)}{T}}}\right) d\lambda$$

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$$+\frac{2\pi T}{v_0}\left[\sum_{j=1}^{n_p^+} p_j^+ + \sum_{j=1}^{n_p^-} p_j^- + \sum_{j=1}^{n_h^+} h_j^+ + \sum_{j=1}^{n_h^-} h_j^-\right] + O(T^2)$$

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Here $\mathcal{Z} = Z(q)$. k_F is the Fermi momentum. In this model $k_F = \pi D$, where D is the average density of the gas.
$v_0 = \frac{\varepsilon_0'}{\mathcal{Z}}$ is the velocity of the sound on the Fermi boundary.

Amplitudes

$$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T \longrightarrow \sum_i e^{-xp[u_i]} B[u_i], \qquad x \to \infty$$

$$B[u] = B_d[u] B_s[u], \quad \text{where} \quad B_d[u] = \exp\left(\int_{\widehat{\mathcal{C}}} \frac{z(\lambda) z(\mu)}{(\lambda - \mu_+)^2} \, d\lambda \, d\mu \right)$$

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In $B_s[u]$ one can set $u(\lambda)=\varepsilon_0(\lambda)+Tu_1(\lambda)$ and $\widehat{s}=\pm q$. Then $B_s[u]$ becomes a constant depending only on ℓ: $B_s[u]=B_s(\ell)$. This constant coincides with the smooth part of the critical form factors of $\mathbf{P}_\ell$ class.

Amplitudes

$$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T \longrightarrow \sum_i e^{-xp[u_i]} B[u_i], \qquad x \to \infty$$

$B_d[u]$ goes to the discrete part of the critical form factors of $\mathbf{P}_\ell$ class up to the replacement $v_0/T \mapsto iL$.

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$$B_d[u] \quad \xrightarrow{T\to 0} \quad C_\ell[Z]\left|\frac{i\pi T}{q\varepsilon_0'}\right|^{2\alpha_\ell^2\mathcal{Z}^2}\left(\frac{\sin\pi\alpha_\ell\mathcal{Z}}{\pi}\right)^{2n}$$

$$\times R_{n_p^+,n_h^+}(\{p^+\},\{h^+\}|\alpha_\ell\mathcal{Z}-\ell)\; R_{n_p^-,n_h^-}(\{p^-\},\{h^-\}|\ell-\alpha_\ell\mathcal{Z})$$

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$$R_{n,m}(\{p\},\{h\}|\nu) = \frac{\prod\limits_{j>k}^{n}(p_j-p_k)^2\;\prod\limits_{j>k}^{m}(h_j-h_k)^2}{\prod\limits_{j=1}^{n}\prod\limits_{k=1}^{m}(p_j+h_k-1)^2}\,\Gamma^2\left(\begin{matrix}\{p_k+\nu\}\;,\;\{h_k-\nu\}\\ \{p_k\}\;,\;\{h_k\}\end{matrix}\right)$$

$$\langle e^{2\pi i\alpha \mathcal{Q}_x}\rangle_T \longrightarrow \sum_i e^{-xp[u_i]} B[u_i], \qquad x \to \infty$$

The sum in the r.h.s. over all possible contours $\widehat{\mathcal{C}}_i$ is equivalent to the sum over integers n, ℓ, $p^{\pm}$ and $h^{\pm}$. On the QTM language this is the sum over the QTM spectrum. In the low-temperature limit this sum turns into the series of critical form factors ($v_0/T \mapsto iL$).

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We know how to sum up such series!

$$\langle e^{2\pi i\alpha\mathcal{Q}_x}\rangle_T \longrightarrow \sum_i e^{-xp[u_i]}B[u_i], \qquad x \to \infty$$

We reproduce the CFT results in the low-temperature limit

$$\langle e^{2i\pi\alpha\mathcal{Q}_x}\rangle_T \longrightarrow \sum_{\ell\in\mathbb{Z}} C(\ell)\cdot e^{2i(\alpha+\ell)k_F x}\cdot\left(\frac{\pi T/v_0}{\sinh\frac{\pi T x}{v_0}}\right)^{2\alpha_\ell^2\mathcal{Z}^2}$$

$$\langle j(x)j(0)\rangle_T \longrightarrow D^2 - \frac{(T\mathcal{Z}/v_0)^2}{2\sinh^2(\pi T x/v_0)} + \sum_{\ell\in\mathbb{Z}^*}\tilde{C}(\ell)\; e^{2ix\ell k_F}\left(\frac{\pi T/v_0}{\sinh(\pi T x/v_0)}\right)^{2\ell^2\mathcal{Z}^2}$$