Form factors in free-fermion models and elliptic determinants

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based on:

arXiv:1012.2856 (N. Iorgov, O.L.)

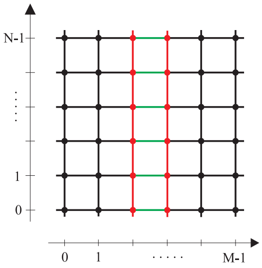
arXiv:1108.3290 (P. Gavrylenko, N. Iorgov, O.L.)

Main examples:

- Spin operator in the Ising model on the square lattice
  - triangular Ising lattice
  - XY chain in a tranverse field
  - BBS_2 model
- $U(1)$ twist fields in the lattice Dirac theory
  - exponential fields of the SG_{ff} model

Aim: compute finite-lattice form factors

- Ising answer is known [A.Bugrij, O.L., '03]
- (involved) proof by separation of variables in [G. von Gehlen, N.Iorgov, S.Pakuliak, V.Shadura, Y.Tykhyy, '06-'08]
- (simple) free-fermionic derivation?



Transfer matrix $V = (2\sinh 2\mathcal{K}_x)^{\frac{N}{2}} V_y^{\frac{1}{2}} V_x V_y^{\frac{1}{2}}$, with

$$V_x = \exp \mathcal{K}_x^* \sum_{j=0}^{N-1} \sigma_j^x, \qquad V_y = \exp \mathcal{K}_y \sum_{j=0}^{N-1} \sigma_j^z \sigma_{j+1}^z$$

- dual couplings: $\sinh 2\mathcal{K}_\alpha \sinh 2\mathcal{K}_\alpha^* = 1$ (ferromagnetic region $\mathcal{K}_x^* < \mathcal{K}_y$)
- symmetries: translation operator T, spin flip $U = \prod_{j=0}^{N-1} \sigma_j^x$
- periodic boundary conditions
- need to find matrix elements of σ_0^z in the basis of common eigenstates of V, T, and U

Jordan-Wigner transformation:

For $j = 0, \ldots, N-1$,

$$p_j = \left(\prod_{k=0}^{j-1} \sigma_k^x\right)\sigma_j^z, \qquad q_j = \left(\prod_{k=0}^{j-1} \sigma_k^x\right)\sigma_j^y$$

satisfy anticommutation relations for Clifford algebra generators:

$$\{p_j, p_k\} = \{q_j, q_k\} = 2\delta_{jk}, \qquad \{p_j, q_k\} = 0.$$

One has

$$V = (2\sinh 2\mathcal{K}_x)^{\frac{N}{2}} \left(\frac{\mathbf{1}+U}{2} V_a + \frac{\mathbf{1}-U}{2} V_p\right),$$

where $V_\nu = V_{y,\nu}^{\frac{1}{2}} V_x V_{y,\nu}^{\frac{1}{2}}$ ($\nu = a, p$) and

$$V_x = \exp\left\{ i\mathcal{K}_x^* \sum_{j=0}^{N-1} p_j q_j \right\}, \qquad V_{y,\nu} = \exp\left\{ -i\mathcal{K}_y \sum_{j=0}^{N-1} p_{j+1} q_j \right\},$$

with $p_N = -p_0$ ($p_N = p_0$) for $\nu = a$ (resp. $\nu = p$). Note that $[V_{a,p}, U] = 0$.

[B.Kaufman, '49]: The conjugation by $V_{a,p}$ acts linearly on the generators $\{p_j\}$ and $\{q_j\}$ (i.e. $V_{a,p}$ belong to Clifford group). By Schur lemma, induced linear transformations fix $V_{a,p}$ up to a scalar multiple.

Consider discrete Fourier transforms

$$p_\theta = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-ij\theta} p_j, \qquad q_\theta = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-ij\theta} q_j.$$

Two sets of quasimomenta will be used:

- $\boldsymbol{\theta}_a = \left\{ \frac{\pi}{N}, \frac{3\pi}{N}, \ldots, 2\pi - \frac{\pi}{N} \right\}$ for V_a,
- $\boldsymbol{\theta}_p = \left\{ 0, \frac{2\pi}{N}, \ldots, 2\pi - \frac{2\pi}{N} \right\}$ for V_p.

Introduce creation-annihilation operators of a- and p-fermions:

$$\begin{cases} 2\psi_\theta^\dagger = e^{-i\theta} \chi_\theta \, p_{-\theta} - i\chi_{-\theta} \, q_{-\theta}, \\ 2\psi_\theta = \ e^{i\theta} \chi_{-\theta} \, p_\theta \, + \, i\chi_\theta \, q_\theta, \end{cases}$$

where

$$\chi_\theta = [\chi_{-\theta}]^{-1} = \left[\frac{\left(1-\alpha e^{i\theta}\right)\left(1-\beta e^{-i\theta}\right)}{\left(1-\beta e^{i\theta}\right)\left(1-\alpha e^{-i\theta}\right)} \right]^{\frac{1}{4}},$$

and $\alpha = \tanh \mathcal{K}_x^* \coth \mathcal{K}_y$, $\beta = \tanh \mathcal{K}_x^* \tanh \mathcal{K}_y$. They satisfy canonical anticommutation relations

$$\{\psi_\theta^\dagger, \psi_{\theta'}\} = \delta_{\theta,\theta'}, \qquad \{\psi_\theta^\dagger, \psi_{\theta'}^\dagger\} = \{\psi_\theta, \psi_{\theta'}\} = 0.$$

$\{\psi_\theta^\dagger\}$, $\{\psi_\theta\}$ transform diagonally under conjugation by V_ν:

$$V_\nu \begin{pmatrix} \psi_\theta^\dagger \\ \psi_\theta \end{pmatrix} V_\nu^{-1} = \begin{pmatrix} e^{-\gamma_\theta} & 0 \\ 0 & e^{\gamma_\theta} \end{pmatrix} \begin{pmatrix} \psi_\theta^\dagger \\ \psi_\theta \end{pmatrix}, \qquad \theta \in \boldsymbol{\theta}_\nu$$

where the function $\gamma_\theta \geq 0$ is defined by

$$\cosh\gamma_\theta = \cosh 2\mathcal{K}_x^* \cosh 2\mathcal{K}_y - \sinh 2\mathcal{K}_x^* \sinh 2\mathcal{K}_y \cos\theta.$$

Taking into account that $\det V_{a,p} = 1$ and $\mathrm{Tr}\, V_{a,p} > 0$, we find that

$$V_\nu = \exp\left\{ -\sum_{\theta\in\boldsymbol{\theta}_\nu} \gamma_\theta \left(\psi_\theta^\dagger\psi_\theta - \frac{1}{2} \right) \right\}, \qquad \nu = a, p.$$

Fock states

$$|\theta_1, \ldots, \theta_k\rangle_\nu = \psi_{\theta_1}^\dagger \ldots \psi_{\theta_k}^\dagger |vac\rangle_\nu, \qquad \theta_1, \ldots, \theta_k \in \boldsymbol{\theta}_\nu,$$

are eigenvectors of V_ν:

$$V_\nu |\theta_1, \ldots, \theta_k\rangle_\nu = \exp\left\{ \frac{1}{2} \sum_{\theta\in\boldsymbol{\theta}_\nu} \gamma_\theta - \sum_{i=1}^{k} \gamma_{\theta_i} \right\} |\theta_1, \ldots, \theta_k\rangle_\nu.$$

Each set of eigenvectors ($\nu = a, p$) constitutes a basis of the space of states.

We reserve the notation $\psi_\theta^\dagger$, ψ_θ for the creation-annihilation operators with $\theta \in \boldsymbol{\theta}_a$ and denote the corresponding operators with $\theta \in \boldsymbol{\theta}_p$ by $\varphi_\theta^\dagger$, φ_θ. Our task is to compute

$$\mathcal{F}^{(l)}_{m,n}\left(\boldsymbol{\theta},\boldsymbol{\theta}'\right) = {}_a\langle\theta_1,\ldots,\theta_m|\sigma_l^z|\theta'_1,\ldots,\theta'_n\rangle_p = \\ = {}_a\langle vac|\psi_{\theta_1}\ldots\psi_{\theta_m}\sigma_l^z\varphi^\dagger_{\theta'_1}\ldots\varphi^\dagger_{\theta'_n}|vac\rangle_p,$$

where $\theta_1,\ldots,\theta_m \in \boldsymbol{\theta}_a$, $\theta'_1,\ldots,\theta'_n \in \boldsymbol{\theta}_p$.

- these are the only nontrivial form factors: since σ_l^z anticommutes with U, its matrix elements between the eigenstates of V of the same type vanish
- periodic boundary conditions $\longrightarrow$ even number of particles in each sector
- it suffices to set $l = 0$ thanks to tranlational invariance

Conjugation by $s_l = \sigma_l^z$ acts linearly on the Clifford algebra generators $\{p_j\}$, $\{q_j\}$ (i.e. σ_l^z also belongs to the Clifford group):

$$\begin{cases} s_l p_j s_l^{-1} = \mathrm{sgn}(l-j)p_j, \\ s_l q_j s_l^{-1} = \mathrm{sgn}(l-1-j)q_j, \end{cases} \qquad j = 0,\ldots,N-1,$$

where $\mathrm{sgn}(x) = 1$ if $x \geq 0$ and -1 if $x < 0$.

Bogoliubov transformations

Let $\{\psi_j^\dagger\}$, $\{\psi_j\}$ be $2N$ fermionic creation-annihilation annihilation operators satisfying canonical ACR. Combine them into N-dimensional column vectors $\boldsymbol{\psi}^\dagger$, $\boldsymbol{\psi}$ (their entries are $2^N \times 2^N$-dimensional matrices). Let σ be a unitary operator such that

$$\sigma \begin{pmatrix} \boldsymbol{\psi}^\dagger \\ \boldsymbol{\psi} \end{pmatrix} \sigma^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}^\dagger \\ \boldsymbol{\psi} \end{pmatrix}$$

Unitarity of σ and ACR imply that $B = \bar{C}$, $A = \bar{D}$ and

$$DC^T + CD^T = 0, \qquad DD^\dagger + CC^\dagger = \mathbf{1}$$

Then [Berezin, '65]

$$\sigma = |\det D|^{\frac{1}{2}} \; : \exp\left\{ -\frac{1}{2}\boldsymbol{\psi}^\dagger D^{-1} C \boldsymbol{\psi}^\dagger + \boldsymbol{\psi}^\dagger (D^{-1} - \mathbf{1}) \boldsymbol{\psi} - \frac{1}{2} \boldsymbol{\psi} B D^{-1} \boldsymbol{\psi} \right\} :$$

so that in particular $\langle \sigma \rangle \stackrel{def}{=} \langle vac|\sigma|vac\rangle = |\det D|^{\frac{1}{2}}$ and

$$\langle vac|\psi_j \psi_k \sigma|vac\rangle = \langle\sigma\rangle \left(D^{-1}C\right)_{jk},$$

$$\langle vac|\psi_j \sigma \psi_k^\dagger|vac\rangle = \langle\sigma\rangle D^{-1}_{jk},$$

$$\langle vac|\sigma \psi_j^\dagger \psi_k^\dagger|vac\rangle = \langle\sigma\rangle \left(BD^{-1}\right)_{jk}.$$

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$$\sigma \begin{pmatrix} \boldsymbol{\varphi}^\dagger \\ \boldsymbol{\varphi} \end{pmatrix} \sigma^{-1} \quad \sigma \begin{pmatrix} \boldsymbol{\psi}^\dagger \\ \boldsymbol{\psi} \end{pmatrix} \sigma^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}^\dagger \\ \boldsymbol{\psi} \end{pmatrix}$$

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so that in particular $\langle \sigma \rangle \stackrel{def}{=} {}_{\psi}\langle vac|\sigma|vac\rangle_{\varphi} = |\det D|^{\frac{1}{2}}$ and

$$\begin{aligned} {}_{\psi}\langle vac|\psi_j \psi_k \sigma|vac\rangle_{\varphi} &= \langle \sigma \rangle \left(D^{-1} C\right)_{jk}, \\ {}_{\psi}\langle vac|\psi_j \sigma \varphi_k^\dagger|vac\rangle_{\varphi} &= \langle \sigma \rangle D^{-1}_{jk}, \\ {}_{\psi}\langle vac|\sigma \varphi_j^\dagger \varphi_k^\dagger|vac\rangle_{\varphi} &= \langle \sigma \rangle \left(B D^{-1}\right)_{jk}. \end{aligned}$$

[Palmer, Hystad, '10]

Proof:

1. $|vac\rangle_{\varphi} \sim \sigma^{-1}\mathcal{O}|vac\rangle_{\psi}$, with $\mathcal{O} = \exp\left\{-\frac{1}{2}\psi^{\dagger}D^{-1}C\psi^{\dagger}\right\}$. Indeed,

$$\begin{aligned}\varphi\sigma^{-1}\mathcal{O}|vac\rangle_{\psi} &= \sigma^{-1}\left(C\psi^{\dagger}+D\psi\right)\mathcal{O}|vac\rangle_{\psi} = \\ &= \sigma^{-1}\mathcal{O}\left(C\psi^{\dagger}+D\mathcal{O}^{-1}\psi\,\mathcal{O}\right)|vac\rangle_{\psi} = \\ &= \sigma^{-1}\mathcal{O}\left(C\psi^{\dagger}+D\left(-D^{-1}C\psi^{\dagger}+\psi\right)\right)|vac\rangle_{\psi} = 0.\end{aligned}$$

2. One has

$$\begin{aligned}&{}_{\psi}\langle vac|\psi_{i_1}\ldots\psi_{i_m}\sigma\varphi^{\dagger}_{j_1}\ldots\varphi^{\dagger}_{j_n}\sigma^{-1}\mathcal{O}|vac\rangle_{\psi} = \\ &={}_{\psi}\langle vac|\psi_{i_1}\ldots\psi_{i_m}(A\psi^{\dagger}+B\psi)_{j_1}\ldots(A\psi^{\dagger}+B\psi)_{j_n}\mathcal{O}|vac\rangle_{\psi} = \\ &={}_{\psi}\langle vac|(-D^{-1}C\psi^{\dagger}+\psi)_{i_1}\ldots(-D^{-1}C\psi^{\dagger}+\psi)_{i_m}\times \\ &\times(D^{-T}\psi^{\dagger}+B\psi)_{j_1}\ldots(D^{-T}\psi^{\dagger}+B\psi)_{j_n}|vac\rangle_{\psi}\end{aligned}$$

3. Wick theorem $\longrightarrow$ Pfaffian of pairings

Multiparticle Ising spin form factors are given by

$$\mathcal{F}^{(l)}_{m,n}\left(\boldsymbol{\theta},\boldsymbol{\theta}'\right)=\left|\det D\right|^{\frac12}\cdot \mathrm{Pf}\, R,$$

where matrix elements of the $(m+n)\times(m+n)$ matrix $R=\begin{pmatrix} R_{\boldsymbol{\theta}\times\boldsymbol{\theta}} & R_{\boldsymbol{\theta}\times\boldsymbol{\theta}'} \\ R_{\boldsymbol{\theta}'\times\boldsymbol{\theta}} & R_{\boldsymbol{\theta}'\times\boldsymbol{\theta}'}\end{pmatrix}$ are defined as

$$\begin{aligned}
\left(R_{\boldsymbol{\theta}\times\boldsymbol{\theta}}\right)_{jk} &= \left(D^{-1}C\right)_{\theta_j,\theta_k}, && j,k=1,\ldots,m,\\
\left(R_{\boldsymbol{\theta}\times\boldsymbol{\theta}'}\right)_{jk} = -\left(R_{\boldsymbol{\theta}'\times\boldsymbol{\theta}}\right)_{kj} &= D^{-1}_{\theta_j,\theta'_k}, && j=1,\ldots,m,\quad k=1,\ldots,n,\\
\left(R_{\boldsymbol{\theta}'\times\boldsymbol{\theta}'}\right)_{jk} &= \left(BD^{-1}\right)_{\theta'_j,\theta'_k}, && j,k=1,\ldots,n.
\end{aligned}$$

Explicit form of the linear transformation induced by spin conjugation:

$$\bar{A}_{\theta,\theta'}=D_{\theta,\theta'}=\frac{e^{-i(l-\frac12)(\theta-\theta')}}{iN}\,\frac{\chi_{-\theta}\chi_{\theta'}+\chi_{\theta}\chi_{-\theta'}}{2\sin\frac{\theta'-\theta}{2}},$$

$$\bar{B}_{\theta,\theta'}=C_{\theta,\theta'}=\frac{e^{-i(l-\frac12)(\theta+\theta')}}{iN}\,\frac{\chi_{\theta}\chi_{\theta'}-\chi_{-\theta}\chi_{-\theta'}}{2\sin\frac{\theta'+\theta}{2}},$$

where $\theta\in\boldsymbol{\theta}_p$, $\theta'\in\boldsymbol{\theta}_a$ and

$$\chi_\theta=\left[\chi_{-\theta}\right]^{-1}=\left[\frac{\left(1-\alpha e^{i\theta}\right)\left(1-\beta e^{-i\theta}\right)}{\left(1-\beta e^{i\theta}\right)\left(1-\alpha e^{-i\theta}\right)}\right]^{\frac14}.$$

Elliptic parametrization of the Ising spectral curve:

The dispersion relation

$$\cosh\gamma_\theta = \cosh 2\mathcal{K}_x^* \cosh 2\mathcal{K}_y - \sinh 2\mathcal{K}_x^* \sinh 2\mathcal{K}_y \cos\theta$$

can be rewritten in terms of $z = e^{i\theta}$, $\lambda = e^{\gamma_\theta}$ as

$$\sinh 2\mathcal{K}_x \frac{\lambda + \lambda^{-1}}{2} + \sinh 2\mathcal{K}_y \frac{z + z^{-1}}{2} = \cosh 2\mathcal{K}_x \cosh 2\mathcal{K}_y .$$

Uniformization of the curve is given by the elliptic functions of modulus $k = \sinh 2\mathcal{K}_x^* / \sinh 2\mathcal{K}_y$:

$$z(u) = \frac{\mathrm{sn}\,(u + i\eta)}{\mathrm{sn}\,(u - i\eta)},$$
$$\lambda(u) = [k\,\mathrm{sn}\,(u + i\eta)\,\mathrm{sn}\,(u - i\eta)]^{-1} .$$

The real parameter η is determined by $\sinh 2\mathcal{K}_x = i\,\mathrm{sn}\,2i\eta$.

These formulas bijectively map the real interval $\mathcal{C}_u = \{u \,|\, \mathrm{Re}\,u \in [-K, K), \mathrm{Im}\,u = 0\}$ to $\mathcal{C}_\theta = \left\{(z, \lambda) = (e^{i\theta}, e^{\gamma_\theta}) |\, \theta \in [0, 2\pi)\right\}$. The inverse image of the point $(e^{i\theta}, e^{\gamma_\theta}) \in \mathcal{C}_\theta$ in $\mathcal{C}_u$ will be denoted by u_θ.

In the elliptic parametrization

$$\frac{\chi_{-\theta}\chi_{\theta'}+\chi_{\theta}\chi_{-\theta'}}{2\sin\frac{\theta'-\theta}{2}}=\frac{\sinh 2\mathcal{K}_y}{\sqrt{\sinh\gamma_\theta \sinh\gamma_{\theta'}}}\,\frac{\mathrm{dn}\,(u_\theta-u_{\theta'})}{\mathrm{sn}\,(u_\theta-u_{\theta'})},$$

$$\frac{\chi_{\theta}\chi_{\theta'}-\chi_{-\theta}\chi_{-\theta'}}{2\sin\frac{\theta'+\theta}{2}}=\frac{-i\sinh 2\mathcal{K}_x^*}{\sqrt{\sinh\gamma_\theta \sinh\gamma_{\theta'}}}\,\mathrm{cn}\,(u_\theta-u_{\theta'})\,.$$

Thus the nontrivial part of the matrix of induced rotation can be encoded into two matrices

$$\Phi_{\theta,\theta'}=\frac{\mathrm{dn}\,(u_\theta-u_{\theta'})}{\mathrm{sn}\,(u_\theta-u_{\theta'})},\qquad \Psi_{\theta,\theta'}=\mathrm{cn}\,(u_\theta-u_{\theta'})\,,\qquad \theta\in\boldsymbol{\theta}_p,\quad \theta'\in\boldsymbol{\theta}_a.$$

- VEV: $\det D \rightsquigarrow \det\Phi$
- 2-particle form factors: $BD^{-1}\rightsquigarrow\Psi\Phi^{-1}$, $D^{-1}\rightsquigarrow\Phi^{-1}$, $D^{-1}C\rightsquigarrow\Phi^{-1}\Psi$
- $\det\Phi$, Φ^{-1} can be found using Frobenius determinant
- the products $\Psi\Phi^{-1}$, $\Phi^{-1}\Psi$ can be computed using a theta functional analog of the Lagrange interpolation identity

Cauchy determinant:
Let $x_1, \ldots, x_N$, $y_1, \ldots, y_N$ be $2N$ indeterminates. The determinant of the Cauchy matrix $V_{ij} = \dfrac{1}{x_i - y_j}$ is given by

$$\det V = \frac{\prod_{i<j}^N (x_i - x_j)(y_j - y_i)}{\prod_{i,j}^N (x_i - y_j)}$$

Frobenius determinant:
Let $x_1, \ldots, x_N$, $y_1, \ldots, y_N$ be $2N$ indeterminates and $\alpha \in \mathbb{C}$. The determinant of the elliptic Cauchy matrix $\tilde{V}_{ij} = \dfrac{\vartheta_1(x_i - y_j + \alpha)}{\vartheta_1(x_i - y_j)\vartheta_1(\alpha)}$ is given by

$$\det \tilde{V} = \frac{\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \alpha\right)}{\vartheta_1(\alpha)} \frac{\prod_{i<j}^N \vartheta_1(x_i - x_j)\vartheta_1(y_j - y_i)}{\prod_{i,j}^N \vartheta_1(x_i - y_j)}$$

N.B. Cofactors of $\tilde{V}$ have the same form, hence the inverse $\tilde{V}^{-1}$ can be easily computed:

$$\tilde{V}_{mn}^{-1} = -\frac{\vartheta_1\left(\sum_{i\neq n}^N x_i - \sum_{i\neq m}^N y_i + \alpha\right)}{\vartheta_1\left(\sum_i^N x_i - \sum_i^N y_i + \alpha\right)\vartheta_1\left(x_n - y_m\right)} \frac{\prod_i^N \vartheta_1\left(x_n - y_i\right)\prod_i^N \vartheta_1\left(y_m - x_i\right)}{\prod_{i\neq n}^N \vartheta_1\left(x_n - x_i\right)\prod_{i\neq m}^N \vartheta_1\left(y_m - y_i\right)}.$$

Theta functions are related to the Jacobi elliptic functions by

$$\mathrm{sn}\, u = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)}, \quad \mathrm{cn}\, u = \frac{\vartheta_4}{\vartheta_2}\frac{\vartheta_2(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)}, \quad \mathrm{dn}\, u = \frac{\vartheta_4}{\vartheta_3}\frac{\vartheta_3(\vartheta_3^{-2}u)}{\vartheta_4(\vartheta_3^{-2}u)},$$

where $\vartheta_i = \vartheta_i(0)$, $i = 2, 3, 4$. Elliptic modulus and half-periods are $k = \vartheta_2^2/\vartheta_3^2$, $2K = \pi\vartheta_3^2$, $2iK' = \pi\tau\vartheta_3^2$.
The functions $\vartheta_{2,3,4}(z)$ can be obtained from $\vartheta_1(z)$ by shifting its argument,

$$\vartheta_2(z) = \vartheta_1\left(z + \frac{\pi}{2}\right), \qquad \vartheta_4(z) = ie^{-i\left(z - \frac{\pi\tau}{4}\right)}\vartheta_1\left(z - \frac{\pi\tau}{2}\right),$$

$$\vartheta_3(z) = e^{-i\left(z - \frac{\pi\tau}{4}\right)}\vartheta_1\left(z + \frac{\pi}{2} - \frac{\pi\tau}{2}\right).$$

Therefore one can deduce from Frobenius formula the determinants and inverses for a number of matrices with entries written in terms of the Jacobi elliptic functions.

Example (Determinant and inverse of Φ)

Let Φ denote a matrix with elements $\Phi_{ij} = \dfrac{\mathrm{dn}(u_i - v_j)}{\mathrm{sn}(u_i - v_j)}$ with $i, j = 1, \ldots, N$. Then

$$\det \Phi = \left(\frac{\vartheta_2\vartheta_4}{\vartheta_3}\right)^N \frac{\vartheta_3\left(\sum_i^N x_i - \sum_i^N y_i\right)}{\vartheta_3} \frac{\prod_{i<j}^N \vartheta_1(x_i - x_j)\vartheta_1(y_j - y_i)}{\prod_{i,j}^N \vartheta_1(x_i - y_j)},$$

$$\Phi^{-1}_{mn} = \frac{\vartheta_3}{\vartheta_2\vartheta_4} \frac{\vartheta_3\left(\sum_{i\neq n}^N x_i - \sum_{i\neq m}^N y_i\right)}{\vartheta_3\left(\sum_i^N x_i - \sum_i^N y_i\right)\vartheta_1(y_m - x_n)} \frac{\prod_i^N \vartheta_1(x_n - y_i)\prod_i^N \vartheta_1(y_m - x_i)}{\prod_{i\neq n}^N \vartheta_1(x_n - x_i)\prod_{i\neq m}^N \vartheta_1(y_m - y_i)},$$

where $x_i = \vartheta_3^{-2}u_i$, $y_i = \vartheta_3^{-2}v_i$.

Theta functional interpolation:

Let $z_1, \ldots, z_M$, $z'_1, \ldots, z'_M$, be complex variables. Identity

$$\sum_i^M \frac{\prod_j^M (z_i - z'_j)}{\prod_{j\neq i}^M (z_i - z_j)} = \sum_i^M z_i - \sum_i^M z'_i,$$

follows from the Lagrange interpolation formula. It has an elliptic analog [Whittaker-Watson]:

If $\{z_i\}$, $\{z'_i\}$ satisfy the balancing condition $\sum_i^M z_i - \sum_i^M z'_i = 0$, then

$$\sum_i^M \frac{\prod_j^M \vartheta_1(z_i - z'_j)}{\prod_{j\neq i}^M \vartheta_1(z_i - z_j)} = 0.$$

Example (matrix products $\Phi^{-1}\Psi$ and $\Psi\Phi^{-1}$)

Let $\Phi_{ij} = \dfrac{\mathrm{dn}(u_i - v_j)}{\mathrm{sn}(u_i - v_j)}$, $\Psi_{ij} = \mathrm{cn}(u_i - v_j)$ with $i, j = 1, \ldots, N$. Then

$$\left(\Psi\Phi^{-1}\right)_{ln} = \frac{\vartheta_3^2}{\vartheta_2^2} \frac{\vartheta_2\left(x_l - x_n + \sum_i^N x_i - \sum_i^N y_i\right)}{\vartheta_3\left(\sum_i^N x_i - \sum_i^N y_i\right)} \prod_i^N \frac{\vartheta_1(x_n - y_i)}{\vartheta_4(x_l - y_i)} \prod_{i\neq n}^N \frac{\vartheta_4(x_l - x_i)}{\vartheta_1(x_n - x_i)},$$

$$\left(\Phi^{-1}\Psi\right)_{ml} = \frac{\vartheta_3^2}{\vartheta_2^2} \frac{\vartheta_2\left(y_m - y_l + \sum_i^N x_i - \sum_i^N y_i\right)}{\vartheta_3\left(\sum_i^N x_i - \sum_i^N y_i\right)} \prod_i^N \frac{\vartheta_1(x_i - y_m)}{\vartheta_4(x_i - y_l)} \prod_{i\neq m}^N \frac{\vartheta_4(y_i - y_l)}{\vartheta_1(y_i - y_m)}.$$

Back to trigonometric parametrization:

From the factorized expressions for $\det\Phi$, Φ^{-1}, $\Psi\Phi^{-1}$, $\Phi^{-1}\Psi$ we get

$$ {}_a\langle vac|s_l|vac\rangle_p = \left[(1-k^2)\prod_{\theta\in\boldsymbol{\theta}_p} e^{\eta_\theta}\prod_{\theta\in\boldsymbol{\theta}_a} e^{-\eta_\theta}\right]^{\frac{1}{8}}, $$

$$ \frac{{}_a\langle\theta|s_l|\theta'\rangle_p}{{}_a\langle vac|s_l|vac\rangle_p} = ie^{-i(l-\frac{1}{2})(\theta-\theta')}\frac{e^{(\eta_\theta-\eta_{\theta'})/2}}{N\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\frac{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}{\sin\frac{\theta-\theta'}{2}}, $$

$$ \frac{{}_a\langle\theta,\theta'|s_l|vac\rangle_p}{{}_a\langle vac|s_l|vac\rangle_p} = -\,ie^{-i(l-\frac{1}{2})(\theta+\theta')}\frac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}\frac{e^{(\eta_\theta+\eta_{\theta'})/2}}{N\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\frac{\sin\frac{\theta-\theta'}{2}}{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}, $$

$$ \frac{{}_a\langle vac|s_l|\theta,\theta'\rangle_p}{{}_a\langle vac|s_l|vac\rangle_p} = -\,ie^{i(l-\frac{1}{2})(\theta+\theta')}\frac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}\frac{e^{-(\eta_\theta+\eta_{\theta'})/2}}{N\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\frac{\sin\frac{\theta-\theta'}{2}}{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}, $$

where $e^{\eta_\theta} = \dfrac{\prod_{\theta'\in\boldsymbol{\theta}_a}\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}{\prod_{\theta'\in\boldsymbol{\theta}_p}\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}$.

N.B. One has

$$ \sqrt{k}\,\mathrm{sn}(u_\theta-u_{\theta'}) = \left[\sqrt{k}\,\mathrm{sn}(u_\theta-u_{\theta'}\pm iK')\right]^{-1} = \left(\frac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}\right)^{\frac{1}{2}}\frac{\sin\frac{\theta-\theta'}{2}}{\sinh\frac{\gamma_\theta+\gamma_{\theta'}}{2}}. $$

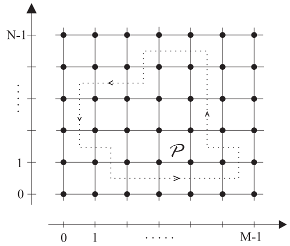
Multiparticle form factors:
From elliptic pfaffian identity

$$\mathrm{Pf}\left(\sqrt{k}\,\mathrm{sn}(z_i - z_j)\right) = \prod_{i<j} \sqrt{k}\,\mathrm{sn}(z_i - z_j)$$

we find

$$\frac{{}_a\langle\theta_1,\ldots,\theta_m|s_l|\theta'_1,\ldots,\theta'_n\rangle_p}{{}_a\langle vac|s_l|vac\rangle_p} = i^{2mn-\frac{m+n}{2}} \prod_{j=1}^{m} \frac{e^{-i(l-\frac{1}{2})\theta_j+\eta_{\theta_j}/2}}{\sqrt{N\sinh\gamma_{\theta_j}}} \prod_{j=1}^{n} \frac{e^{i(l-\frac{1}{2})\theta'_j-\eta_{\theta'_j}/2}}{\sqrt{N\sinh\gamma_{\theta'_j}}} \times$$

$$\times\left(\frac{\sinh 2\mathcal{K}_y}{\sinh 2\mathcal{K}_x}\right)^{\frac{(m-n)^2}{4}} \prod_{1\leq i<j\leq m} \frac{\sin\frac{\theta_i-\theta_j}{2}}{\sinh\frac{\gamma_{\theta_i}+\gamma_{\theta_j}}{2}} \prod_{1\leq i<j\leq n} \frac{\sin\frac{\theta'_i-\theta'_j}{2}}{\sinh\frac{\gamma_{\theta'_i}+\gamma_{\theta'_j}}{2}} \prod_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \frac{\sinh\frac{\gamma_{\theta_i}+\gamma_{\theta'_j}}{2}}{\sin\frac{\theta_i-\theta'_j}{2}}.$$



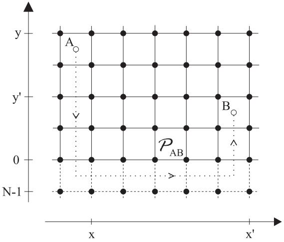
Lattice Dirac action $S[\psi, \bar{\psi}] = \bar{\psi} D \psi$ with

$$D = \frac{1}{c_x^*} \begin{pmatrix} c_x^* s_y - s_x^* c_y \nabla_y & -c_y + c_x^* \nabla_x \\ c_y - c_x^* \nabla_{-x} & c_x^* s_y - s_x^* c_y \nabla_{-y} \end{pmatrix}$$

- boundary conditions

$$\begin{cases} \psi_{x+M,y} = -\psi_{x,y}, \\ \bar{\psi}_{x+M,y} = -\bar{\psi}_{x,y}, \end{cases} \qquad \begin{cases} \psi_{x,y+N} = e^{2\pi i \alpha} \psi_{x,y}, \\ \bar{\psi}_{x,y+N} = e^{-2\pi i \alpha} \bar{\psi}_{x,y}. \end{cases}$$

- parameters $c_i = \cosh \mathcal{K}_i$, $c_i^* = \cosh \mathcal{K}_i$, $s_i = \sinh \mathcal{K}_i$, $s_i^* = \sinh \mathcal{K}_i^*$ $(i = x, y)$



- $\delta S_{AB} = \delta S_A + \delta S_B + \delta S_{\mathrm{b.c.}}$ with

$$\delta S_A = 2i \sin \pi\nu \sum_{y''=0}^{y-1} \left(e^{i\pi\nu} \bar{\psi}^1_{x-1,y''} \psi^2_{x,y''} + e^{-i\pi\nu} \bar{\psi}^2_{x,y''} \psi^1_{x-1,y''} \right)$$

- $\delta S_{\mathrm{b.c.}} \rightsquigarrow$ change of boundary conditions from α- to α'-periodic ($\alpha' = \alpha + \nu$)
- two-point function of $U(1)$ twist fields

$$\langle \mathcal{O}_{\alpha,\alpha'}(A) \mathcal{O}_{\alpha',\alpha}(B) \rangle = \frac{\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\ e^{S[\psi,\bar{\psi}]+\delta S_{AB}}}{\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\ e^{S[\psi,\bar{\psi}]}}.$$

- in the heuristic continuum limit, $\mathcal{O}(A)$ corresponds to integrating over field configurations with counterclockwise monodromy $e^{2\pi i\nu}$ of ψ around A

Canonical formalism

- $4N$ fermionic creation-annihilation operators $\{a_y\}$, $\{a_y^\dagger\}$, $\{b_y\}$, $\{b_y^\dagger\}$ $(y = 0, \ldots, N-1)$
- transition can be carried out using coherent states
- α-periodic transfer matrix

$$V_\alpha =: \exp \sum_{y=0}^{N-1} \left\{ b_y^\dagger (s_y - \frac{c_y}{c_x} \nabla_{-y}) a_y^\dagger + (\frac{c_y}{c_x^*} - 1) \left(a_y^\dagger a_y + b_y^\dagger b_y \right) + a_y (s_y - \frac{c_y}{c_x} \nabla_y) b_y \right\} :$$

$$\text{where } \begin{pmatrix} a_{y+N}^\dagger \\ b_{y+N} \end{pmatrix} = e^{2\pi i \alpha} \begin{pmatrix} a_y^\dagger \\ b_y \end{pmatrix}, \begin{pmatrix} a_{y+N} \\ b_{y+N}^\dagger \end{pmatrix} = e^{-2\pi i \alpha} \begin{pmatrix} a_y \\ b_y^\dagger \end{pmatrix}.$$

- operator of translations

$$T_\alpha =: \exp \sum_{y=0}^{N-1} \left(a_y^\dagger a_{y+1} - a_y^\dagger a_y + b_y^\dagger b_{y+1} - b_y^\dagger b_y \right) :$$

- $U(1)$ charge $Q = \sum_{y=0}^{N-1} \left(a_y^\dagger a_y - b_y^\dagger b_y \right)$
- twist field

$$\mathcal{O}_{\alpha,\alpha'}(y^*) = T_\alpha^{-y} T_{\alpha'}^{y} = \exp 2\pi i \nu \sum_{y'=0}^{y-1} \left(-a_{y'}^\dagger a_{y'} + b_{y'}^\dagger b_{y'} \right)$$

VEV and 2-particle form factors:

$$_\alpha\langle vac|\mathcal{O}_{\alpha,\alpha'}(0^*)|vac\rangle_{\alpha'} = \frac{\vartheta_3(X_{\alpha'}-X_\alpha)}{\vartheta_3}\left[\prod_{\theta\in\boldsymbol{\theta}_\alpha} e^{-\eta_\theta}\prod_{\theta\in\boldsymbol{\theta}_{\alpha'}} e^{\eta_\theta}\right]^{\frac{1}{4}},$$

$$\frac{{}^{\pm}_{\alpha}\langle\theta|\mathcal{O}_{\alpha,\alpha'}(0^*)|\theta'\rangle^{\pm}_{\alpha'}}{_\alpha\langle vac|\mathcal{O}_{\alpha,\alpha'}(0^*)|vac\rangle_{\alpha'}} = -\,e^{\pm i(\theta'-\theta-2\pi\nu)/2}\times$$

$$\times\frac{s_y\sin\pi\nu\, e^{(\eta_\theta-\eta_{\theta'})/2}}{N\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\,\frac{\vartheta_2\vartheta_4}{\vartheta_3}\,\frac{\vartheta_3(X_{\alpha'}-X_\alpha+x_\theta-x_{\theta'})}{\vartheta_3(X_{\alpha'}-X_\alpha)\vartheta_1(x_\theta-x_{\theta'})},$$

$$\frac{{}^{+-}_{\alpha}\langle\theta,\theta'|\mathcal{O}_{\alpha,\alpha'}(0^*)|vac\rangle_{\alpha'}}{_\alpha\langle vac|\mathcal{O}_{\alpha,\alpha'}(0^*)|vac\rangle_{\alpha'}} = -\,e^{i(\theta'-\theta)/2}\times$$

$$\times\frac{is_y\sin\pi\nu\, e^{(\eta_\theta+\eta_{\theta'})/2}}{N\sqrt{\sinh\gamma_\theta\sinh\gamma_{\theta'}}}\,\frac{\vartheta_2\vartheta_4}{\vartheta_3}\,\frac{\vartheta_2(X_{\alpha'}-X_\alpha+x_\theta+x_{\theta'})}{\vartheta_3(X_{\alpha'}-X_\alpha)\vartheta_4(x_\theta+x_{\theta'})},$$

with $X_\alpha=\sum_{\theta\in\boldsymbol{\theta}_\alpha}x_\theta$ and

$$\eta_\theta = 2\ln\frac{\prod_{\theta'\in\boldsymbol{\theta}_{\alpha'}}\vartheta_4(x_{\theta'})\prod_{\theta'\in\boldsymbol{\theta}_\alpha}\vartheta_4(x_\theta+x_{\theta'})}{\prod_{\theta'\in\boldsymbol{\theta}_\alpha}\vartheta_4(x_{\theta'})\prod_{\theta'\in\boldsymbol{\theta}_{\alpha'}}\vartheta_4(x_\theta+x_{\theta'})} - \ln\frac{\prod_{\theta'\in\boldsymbol{\theta}_{\alpha'}}\sinh\frac{\gamma_\pi+\gamma_\theta}{2}}{\prod_{\theta'\in\boldsymbol{\theta}_\alpha}\sinh\frac{\gamma_\pi+\gamma_\theta}{2}}$$

In the thermodynamic limit $\eta_\theta\to 0$, $X_{\alpha'}-X_\alpha\to\pi\nu$ so that e.g.

$$\langle\mathcal{O}_\nu\rangle = \frac{\vartheta_3(\pi\nu|\tau)}{\vartheta_3(0|\tau)} \quad\Rightarrow\quad \text{scaling dimension } \nu^2 \text{ for } |\nu|\le\frac{1}{2}.$$

Scaling limit:
Gap in the energy spectrum closes as $k \to 1$ ($\tau \to i0$).
Set $\varepsilon = \frac{1-k^2}{2c_y}$, then in the scaling limit

$$\varepsilon \to 0, \qquad \theta \to \varepsilon \sinh \xi, \qquad \gamma_\theta \to \varepsilon s_y \cosh \xi$$

Scaled normalized 2-particle form factors:

$$\mathbb{F}_\nu(\xi;|\xi';) = \lim_{\varepsilon \to 0} \varepsilon \frac{{}^{+}\langle \varepsilon \sinh \xi | \mathcal{O}_\nu | \varepsilon \sinh \xi' \rangle^{+}}{\langle vac | \mathcal{O}_\nu | vac \rangle},$$

$$\mathbb{F}_\nu(\xi;\xi'|;) = \lim_{\varepsilon \to 0} \varepsilon \frac{{}^{+-}\langle \varepsilon \sinh \xi, -\varepsilon \sinh \xi' | \mathcal{O}_\nu | vac \rangle}{\langle vac | \mathcal{O}_\nu | vac \rangle}.$$

- modular transformation $\tau \to -\frac{1}{\tau}$
- triple product formula with $q \to 0$
- reproduce form factors of the exponential fields in the $\mathrm{SG_{ff}}$-theory:

$$\mathbb{F}_\nu(\xi;|\xi';) = \frac{\sin \pi\nu}{\sqrt{\cosh \xi \cosh \xi'}} \frac{e^{\nu(\xi'-\xi-i\pi)}}{\sinh \frac{\xi'-\xi}{2}},$$

$$\mathbb{F}_\nu(\xi;\xi'|;) = \frac{\sin \pi\nu}{\sqrt{\cosh \xi \cosh \xi'}} \frac{ie^{\nu(\xi'-\xi)}}{\cosh \frac{\xi'-\xi}{2}}.$$

Summary

Factorized expressions for finite-lattice VEVs and form factors of Ising spin and $U(1)$ twist fields in the Dirac theory have been derived using:

- generalized Bogoliubov transformations
- Frobenius determinant and theta functional interpolation

THANK YOU FOR YOUR ATTENTION !