



Factorizing F -matrices – a diagrammatic approach

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Outline

- 1 Introduction
- 2 Definitions and statement of theorem
- 3 Lemmas
- 4 Proof of theorem
- 5 Discussion



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- I present a new expression for the $sl(n)$ F -matrix
- it is equivalent to the expression of Albert, Boos, Flume and Ruhlig
- this expression is in terms of partial F -matrices à la Maillet and Sanchez de Santos
- I present an easy proof of the factorizing property using this expression
- I use diagrammatic tensor notation throughout



Definitions and statement of theorem

- several definitions are presented which lead up the definition of the F -matrix



Definitions and statement of theorem

- several definitions are presented which lead up the definition of the F -matrix
- the main result is stated



$$(R_{12})_{i_1 i_2}^{j_1 j_2} = \begin{array}{c} j_2 \quad j_1 \\ \diagdown \quad \diagup \\ i_1 \quad i_2 \end{array} = \begin{cases} a(\lambda_1, \lambda_2), & i_1 = i_2 = j_1 = j_2 \\ b(\lambda_1, \lambda_2), & i_1 = j_1, i_2 = j_2, i_1 \neq i_2 \\ c_+(\lambda_1, \lambda_2), & i_1 = j_2, i_2 = j_1, i_1 < i_2 \\ c_-(\lambda_1, \lambda_2), & i_1 = j_2, i_2 = j_1, i_1 > i_2 \\ 0, & \text{otherwise} \end{cases}$$



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- where

$$a(\lambda, \mu) = 1, b(\lambda, \mu) = \frac{\lambda - \mu}{\lambda - \mu + \eta}, c_{\pm}(\lambda, \mu) = \frac{\eta}{\lambda - \mu + \eta}, (\text{XXX})$$



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$$a(\lambda, \mu) = 1, b(\lambda, \mu) = \frac{\sinh(\lambda - \mu)}{\sinh(\lambda - \mu + \eta)}, c_{\pm}(\lambda, \mu) = \frac{e^{\pm(\lambda - \mu)} \sinh(\eta)}{\sinh(\lambda - \mu + \eta)}, \text{ (XXZ)}$$



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- draw a bipartite graph of σ
- each intersection is an R -matrix
- joining an arm to a leg corresponds to contraction
- any graph is equivalent via Y-B & unitarity



$$(I_{12})_{i_1 i_2}^{j_1 j_2} = \begin{array}{c} j_2 \quad j_1 \\ \diagdown \quad / \\ \cdot \\ / \quad \diagdown \\ i_1 \quad i_2 \end{array} = \begin{cases} 1, & \text{if } i_1 = j_1 \text{ and } i_2 = j_2 \\ 0, & \text{otherwise} \end{cases}$$



$$(R_{12}^k)_{i_1 i_2}^{j_1 j_2} = \begin{array}{c} j_2 \quad j_1 \\ \diagdown \quad \diagup \\ i_1 \quad i_2 \end{array} \text{ tier } k = \begin{cases} \begin{array}{c} \diagdown \quad \diagup \\ i_1 \quad i_2 \end{array}, & \text{if } i_1 \leq k \text{ and } i_2 \leq k \\ \begin{array}{c} \diagup \quad \diagdown \\ i_1 \quad i_2 \end{array}, & \text{if } i_1 > k \text{ or } i_2 > k \end{cases}$$



$$(R_{1\dots(N-1),N}^k)_{i_1\dots i_N}^{j_1\dots j_N} = \text{diagram} \quad \text{tier } k$$

The diagram is a string of N vertical lines representing a partial F -matrix. The top of each line is labeled with a column index j_i and the bottom with a row index i_i . The lines are connected by a horizontal line at a certain level, labeled "tier k ". The connections are as follows:

- Line j_N is connected to line i_{N-1} .
- Line j_{N-1} is connected to line i_{N-2} .
- Line j_{N-2} is connected to line i_{N-2} .
- Line j_2 is connected to line i_2 .
- Line j_1 is connected to line i_1 .
- Line j_1 is also connected to line i_N .



$$(R_{1\dots(N-1),N}^k)_{i_1\dots i_N}^{j_1\dots j_N} = \text{diagram}$$

The diagram shows a sequence of $(N-1)$ tier- k R -matrices. Each matrix is represented by a horizontal line with N vertical lines extending upwards and downwards. The top labels are $j_N, j_{N-1}, j_{N-2}, \dots, j_2, j_1$ and the bottom labels are $i_{N-1}, i_{N-2}, \dots, i_2, i_1, i_N$. The first matrix has a curved line connecting its top-left and bottom-right vertices. The last matrix has a curved line connecting its top-right and bottom-left vertices. The text "tier k " is placed to the right of the diagram.

- this is a string of $(N - 1)$ tier- k R -matrices contracted with each other



the tier- k partial F -matrix

$$\left(I_{1 \dots (N-1), N} \right)_{i_1 \dots i_N}^{j_1 \dots j_N} = \text{Diagram}$$

The diagram consists of a horizontal line with several vertical lines crossing it. From left to right, there are vertical lines labeled i_{N-1} , i_{N-2} , i_2 , i_1 , and i_N . Above the horizontal line, there are labels j_N , j_{N-1} , j_{N-2} , j_2 , and j_1 . The diagram shows a sequence of crossings: the line from j_N crosses over i_{N-1} , the line from j_{N-1} crosses over i_{N-2} , and so on, ending with the line from j_1 crossing over i_N . Ellipses between j_{N-2} and j_2 indicate intermediate crossings.

the tier- k partial F -matrix

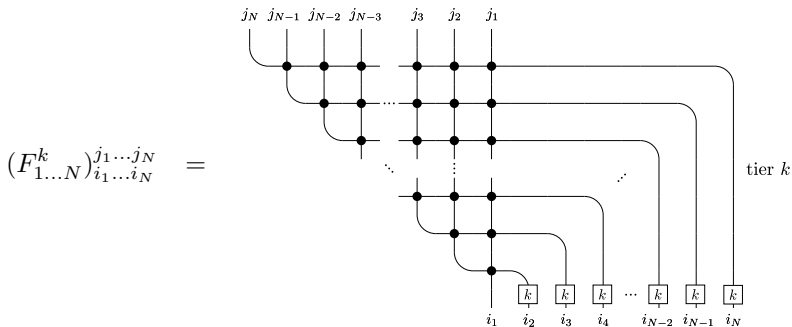
$$\begin{aligned}
 (F_{1\dots(N-1),N}^k)^{j_1\dots j_N}_{i_1\dots i_N} &= \text{Diagram with } N \text{ vertical lines labeled } j_N, j_{N-1}, j_{N-2}, \dots, j_2, j_1 \text{ at the top and } i_{N-1}, i_{N-2}, \dots, i_2, i_1, i_N \text{ at the bottom. A horizontal line connects the lines, with a box labeled } k \text{ on the right side. The label 'tier } k' \text{ is to the right of the box.} \\
 &= \left\{ \begin{array}{l} \text{Diagram with } N \text{ vertical lines and a horizontal line connecting them, with a box labeled } i_N \text{ on the right side.} \\ \text{Diagram with } N \text{ vertical lines and a horizontal line connecting them, with a box labeled 'tier } k' \text{ on the right side.} \end{array} \right. , \quad \begin{array}{l} \text{if } i_N = k, \\ \text{if } i_N \neq k. \end{array}
 \end{aligned}$$



$$F_{1\dots N}^k = F_{1,2}^k F_{12,3}^k \cdots F_{1\dots(N-1),N}^k$$



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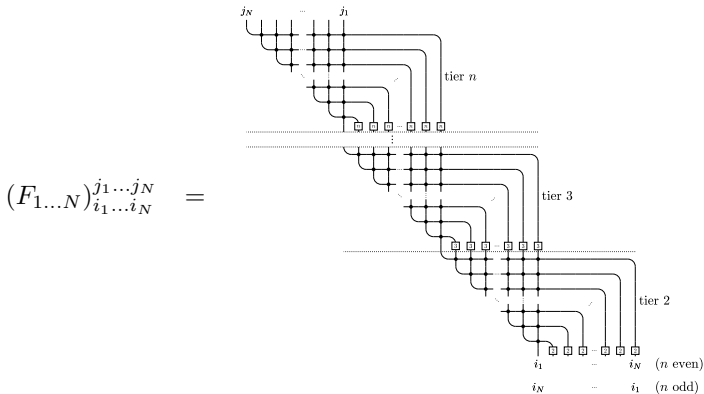




$$F_{1\dots N} = \begin{cases} F_{1\dots N}^2 F_{N\dots 1}^3 \cdots F_{1\dots N}^n, & n \text{ even} \\ F_{N\dots 1}^2 F_{1\dots N}^3 \cdots F_{1\dots N}^n, & n \text{ odd} \end{cases}$$



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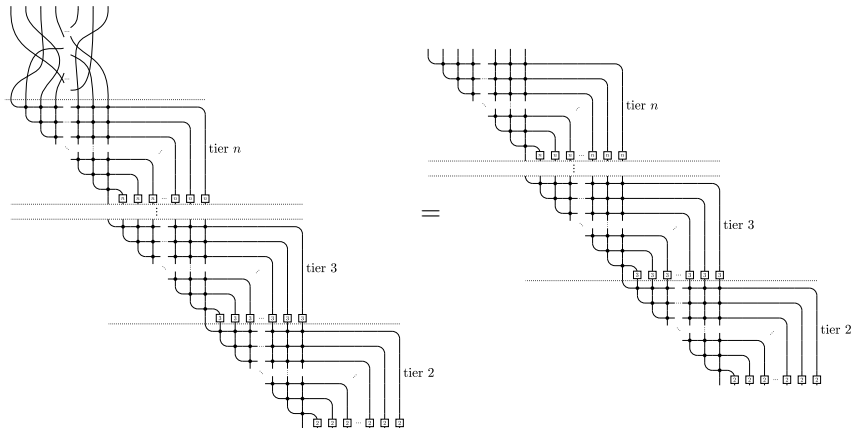




$$F_{\sigma(1)\dots\sigma(N)}R^\sigma = F_{1\dots N}$$



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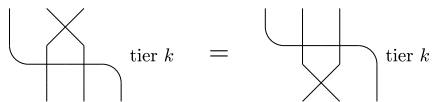
Lemmas

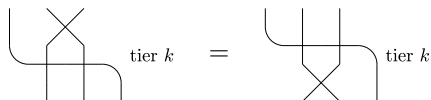
- tier- k Yang-Baxter equation
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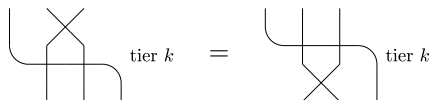
- tier- k Yang-Baxter equation
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- Lemma 4, for passing a tier- k R -matrix through an F -matrix





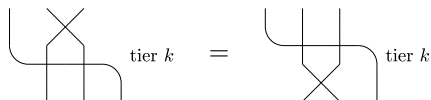
Proof. Two cases:

- all colors $\leq k$
 - all R -matrices, so true by standard Yang-Baxter



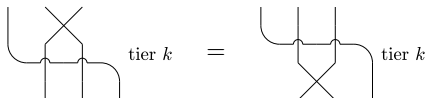
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- all colors $\leq k$
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- one or more colors $> k$
 - at least two of the vertices are identity matrices and the statement holds
 - for example:

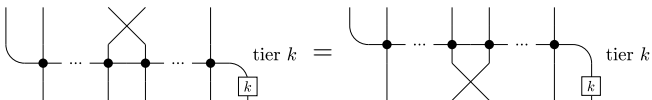




$$F_{1\dots(i+1)i\dots(N-1),N}^k R_{i(i+1)}^k = R_{i(i+1)}^k F_{1\dots(N-1),N}^k$$

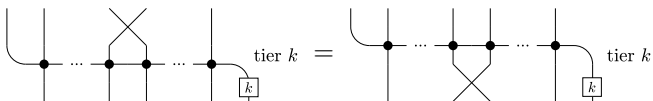


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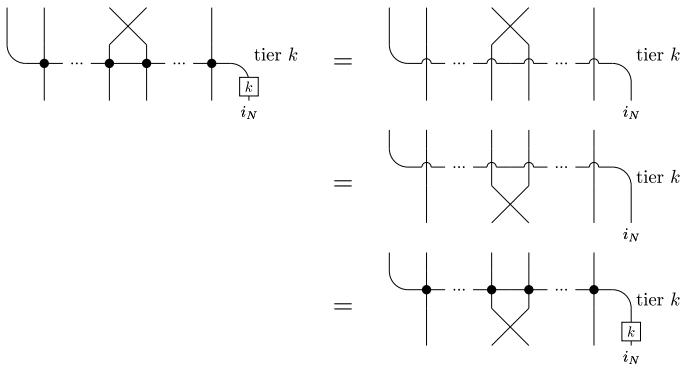
Proof. Two cases:

Case 1: $i_N = k$

Case 2: $i_N \neq k$

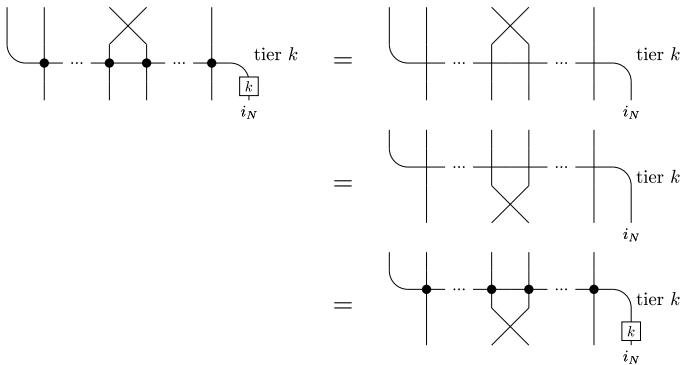


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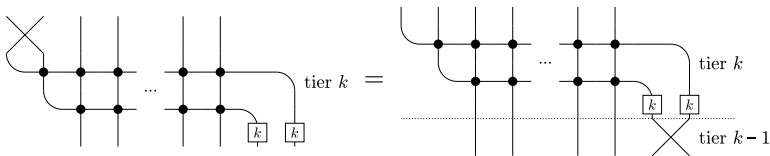




$$\begin{aligned}
 & F_{1\dots(N-2),N}^k F_{1\dots(N-2)N,(N-1)}^k R_{(N-1)N}^k \\
 &= R_{N(N-1)}^{k-1} F_{1\dots(N-2),(N-1)}^k F_{1\dots(N-1),N}^k
 \end{aligned}$$

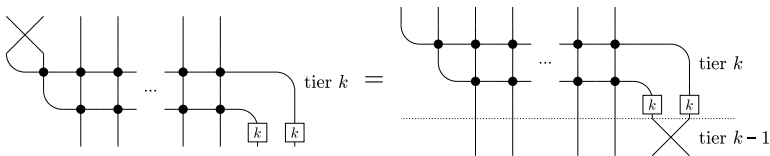


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Proof. Four cases:

Case 1: $i_{N-1} = k, \quad i_N = k$

Case 2: $i_{N-1} \neq k, \quad i_N \neq k$

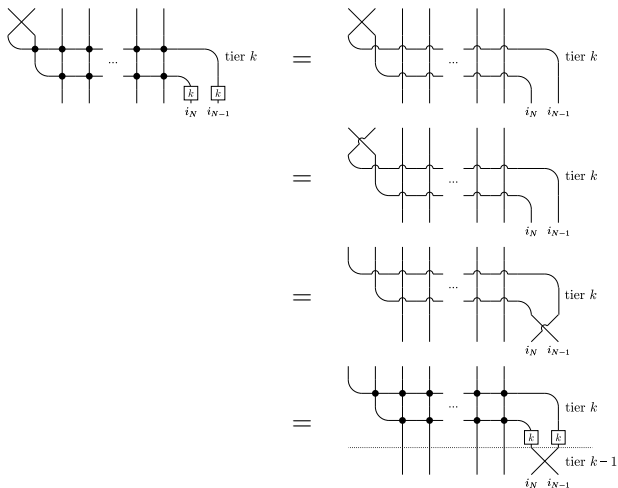
Case 3: $i_{N-1} \neq k, \quad i_N = k$

Case 4: $i_{N-1} = k, \quad i_N \neq k$



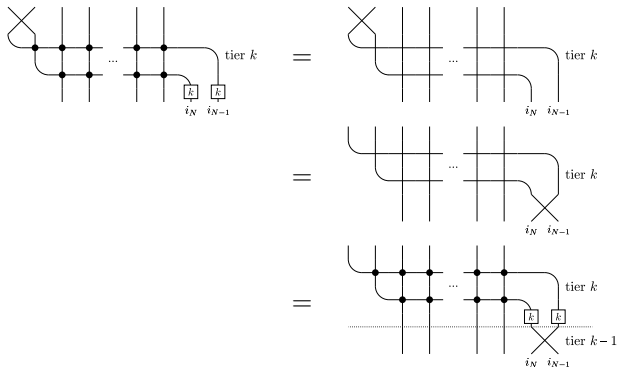
lemma 2

Case 1, $i_{(N-1)} = k, i_N = k$:



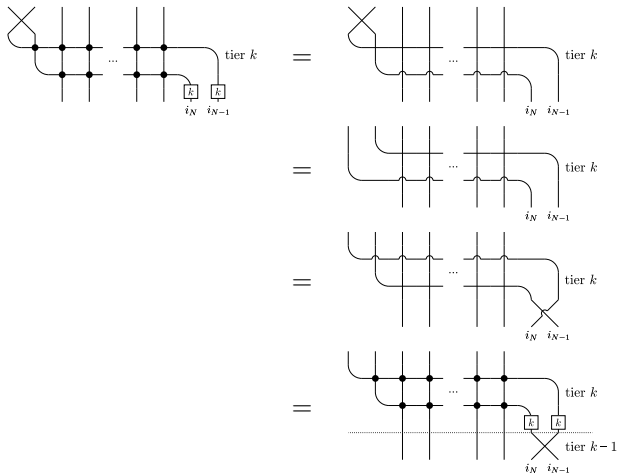


Case 2, $i_{(N-1)} \neq k, i_N \neq k$:



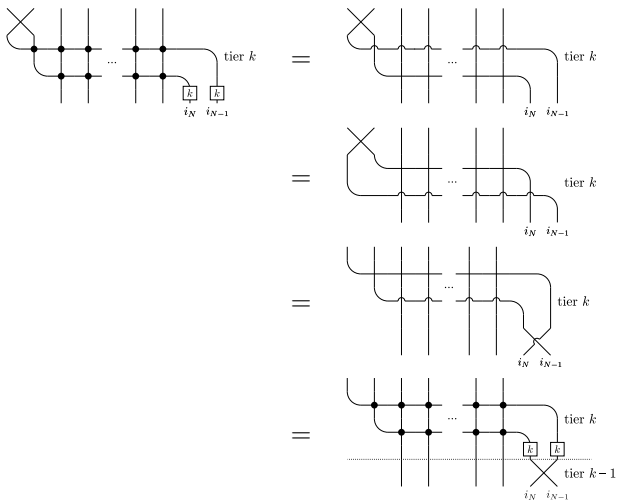


Case 3, $i_{N-1} \neq k, i_N = k$:





Case 4, $i_{N-1} = k, i_N \neq k$:

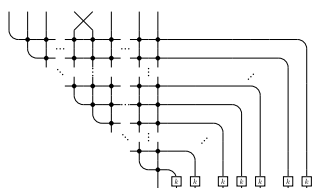
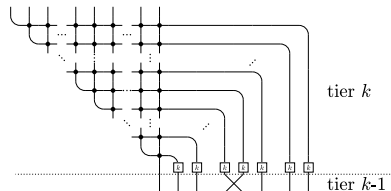




$$F_{1\dots(i+1)i\dots N}^k R_{i(i+1)}^k = R_{(i+1)i}^{k-1} F_{1\dots N}^k$$

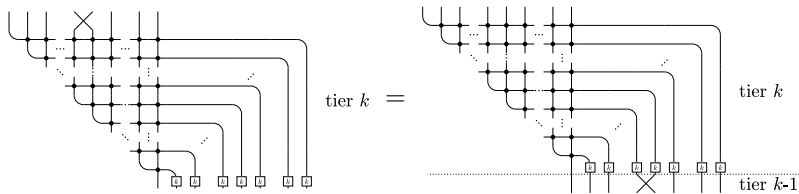


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tier k =tier k tier $k-1$



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Proof. Obvious!

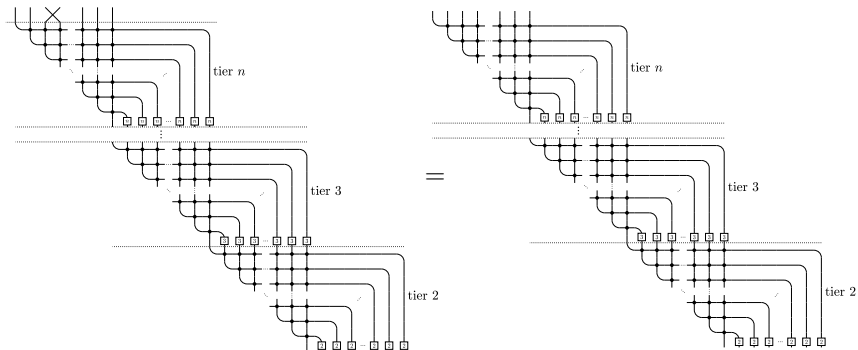


$$\begin{aligned}
 F_{1\dots(i+1)i\dots N} R_{i(i+1)} &= \begin{cases} I_{(i+1)i} F_{1\dots N}, & n \text{ even} \\ I_{i(i+1)} F_{1\dots N}, & n \text{ odd} \end{cases} \\
 &= F_{1\dots N}
 \end{aligned}$$



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- $R_{12}^n = R_{12}$ and $R_{12}^1 = I_{12}$



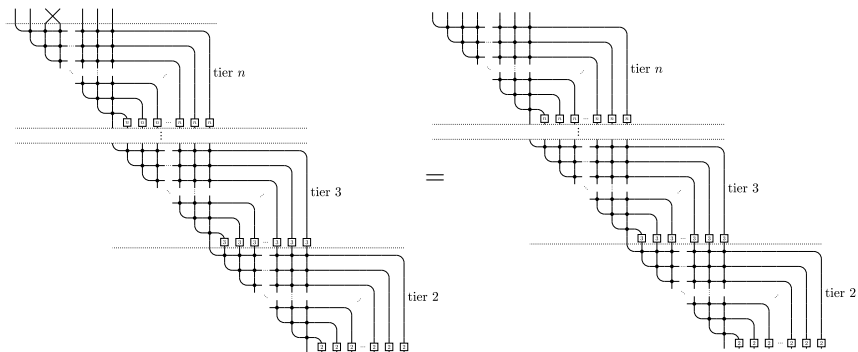
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- then ...



Proof. Observe that

- $R_{12}^n = R_{12}$ and $R_{12}^1 = I_{12}$
- then ... obvious!

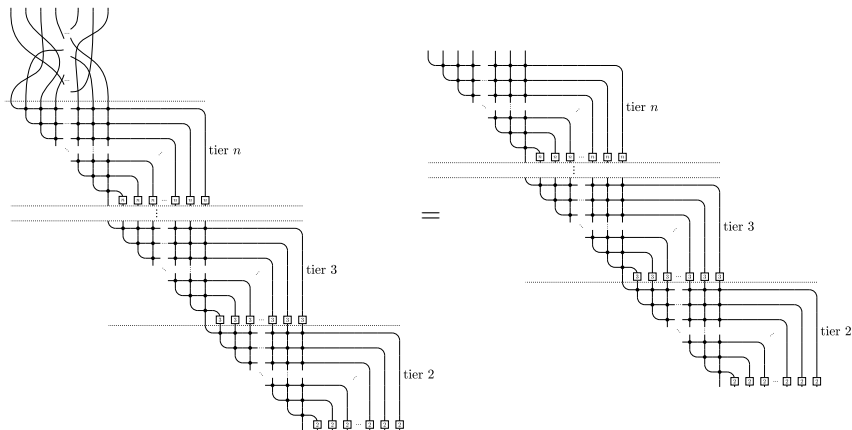




$$F_{\sigma(1)\dots\sigma(N)}R^\sigma = F_{1\dots N}$$

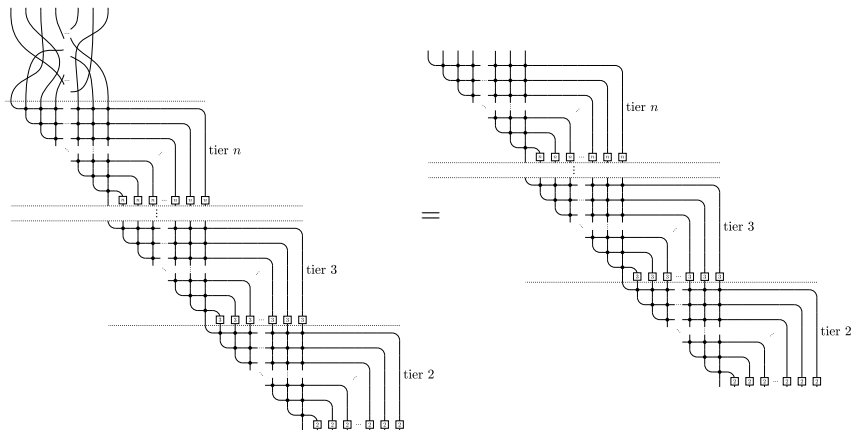


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Proof. Obvious!



Discussion

Questions and comments.