Factorizing F-matrices – a diagrammatic approach

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with Michael Wheeler

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Outline

Introduction

- I present a new expression for the $sl(n)$ F-matrix

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- it is equivalent to the expression of Albert, Boos, Flume and Ruhlig
- this expression is in terms of partial F-matrices à la Maillet and Sanchez de Santos
- I present an easy proof of the factorizing property using this expression
- I use diagrammatic tensor notation throughout

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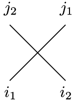
- the main result is stated

$$(R_{12})^{j_1 j_2}_{i_1 i_2} =$$



$$= \begin{cases} a(\lambda_1, \lambda_2), & i_1 = i_2 = j_1 = j_2 \\ b(\lambda_1, \lambda_2), & i_1 = j_1, i_2 = j_2, i_1 \neq i_2 \\ c_+(\lambda_1, \lambda_2), & i_1 = j_2, i_2 = j_1, i_1 < i_2 \\ c_-(\lambda_1, \lambda_2), & i_1 = j_2, i_2 = j_1, i_1 > i_2 \\ 0, & \text{otherwise} \end{cases}$$

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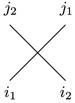

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- the object in the middle represents the components of the R-matrix in diagrammatic tensor notation

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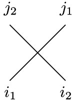

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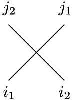
- where $a(\lambda, \mu) = 1, b(\lambda, \mu) = \frac{\lambda - \mu}{\lambda - \mu + \eta}, c_\pm(\lambda, \mu) = \frac{\eta}{\lambda - \mu + \eta}$, (XXX)

$$(R_{12})^{j_1 j_2}_{i_1 i_2} \quad = \quad$$

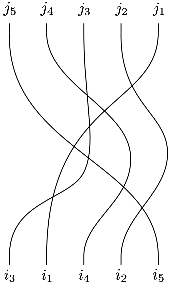


$$= \begin{cases} a(\lambda_1,\lambda_2), & i_1 = i_2 = j_1 = j_2 \\ b(\lambda_1,\lambda_2), & i_1 = j_1, i_2 = j_2, i_1 \neq i_2 \\ c_+(\lambda_1,\lambda_2), & i_1 = j_2, i_2 = j_1, i_1 < i_2 \\ c_-(\lambda_1,\lambda_2), & i_1 = j_2, i_2 = j_1, i_1 > i_2 \\ 0, & \text{otherwise} \end{cases}$$

- the object in the middle represents the components of the R-matrix in diagrammatic tensor notation
- where
  $a(\lambda,\mu) = 1, b(\lambda,\mu) = \frac{\lambda-\mu}{\lambda-\mu+\eta}, c_\pm(\lambda,\mu) = \frac{\eta}{\lambda-\mu+\eta}$, (XXX) or
  $a(\lambda,\mu) = 1, b(\lambda,\mu) = \frac{\sinh(\lambda-\mu)}{\sinh(\lambda-\mu+\eta)}, c_\pm(\lambda,\mu) = \frac{e^{\pm(\lambda-\mu)}\sinh(\eta)}{\sinh(\lambda-\mu+\eta)}$, (XXZ)

the tier-k R-matrix

$$(R^\sigma)^{j_1\ldots j_N}_{i_1\ldots i_N} =$$



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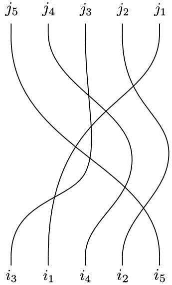
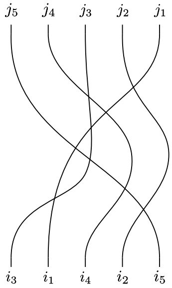


- draw a bipartite graph of σ

$$(R^{\sigma})^{j_1\dots j_N}_{i_1\dots i_N} =$$



- draw a bipartite graph of σ
- each intersection is an R-matrix

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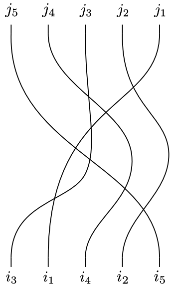

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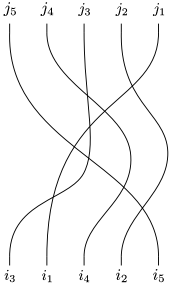
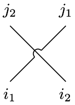
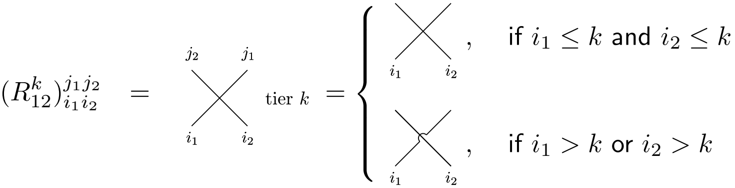


- draw a bipartite graph of σ
- each intersection is an R-matrix
- joining an arm to a leg corresponds to contraction
- any graph is equivalent via Y-B & unitarity

$$(I_{12})^{j_1 j_2}_{i_1 i_2} =$$



$$= \begin{cases} 1, & \text{if } i_1 = j_1 \text{ and } i_2 = j_2 \\ 0, & \text{otherwise} \end{cases}$$

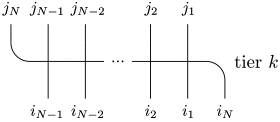


$(R^k_{12})^{j_1 j_2}_{i_1 i_2} =$ tier k $=$

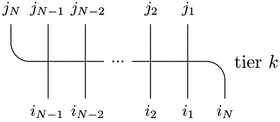
, if $i_1 \leq k$ and $i_2 \leq k$

, if $i_1 > k$ or $i_2 > k$

$$\left(R^{k}_{1\ldots(N-1),N}\right)^{j_1\ldots j_N}_{i_1\ldots i_N} =$$

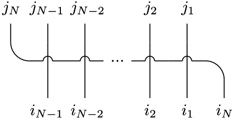


$$(R^{k}_{1\ldots(N-1),N})^{j_1\ldots j_N}_{i_1\ldots i_N} \quad =$$



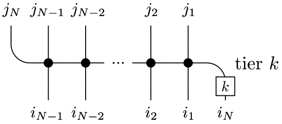
- this is a string of $(N-1)$ tier-k R-matrices contracted with each other

$$\left(I_{1\ldots(N-1),N}\right)^{j_1\ldots j_N}_{i_1\ldots i_N} =$$

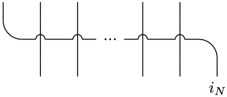


the tier-k partial F-matrix

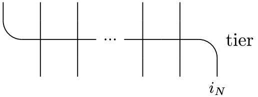
$(F^{k}_{1\ldots(N-1),N})^{j_1\ldots j_N}_{i_1\ldots i_N} \quad =$



$=$



, if $i_N = k$,

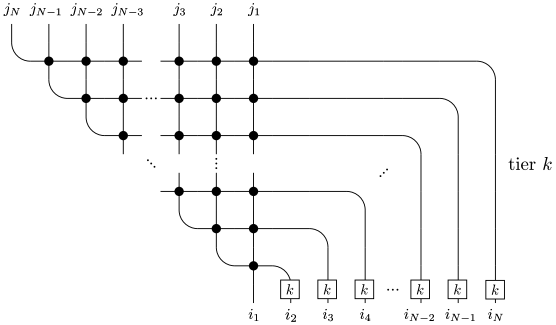


, if $i_N \neq k$.

$$F^{k}_{1\ldots N} \quad = \quad F^{k}_{1,2} F^{k}_{12,3} \cdots F^{k}_{1\ldots(N-1),N}$$

$$F^k_{1\dots N} \quad = \quad F^k_{1,2}F^k_{12,3}\dots F^k_{1\dots(N-1),N}$$

$$\left(F^k_{1\dots N}\right)^{j_1\dots j_N}_{i_1\dots i_N} \quad =$$



the F matrix

$$F_{1\dots N} = \begin{cases} F^{2}_{1\dots N} F^{3}_{N\dots 1} \dots F^{n}_{1\dots N}, & n \text{ even} \\ F^{2}_{N\dots 1} F^{3}_{1\dots N} \dots F^{n}_{1\dots N}, & n \text{ odd} \end{cases}$$

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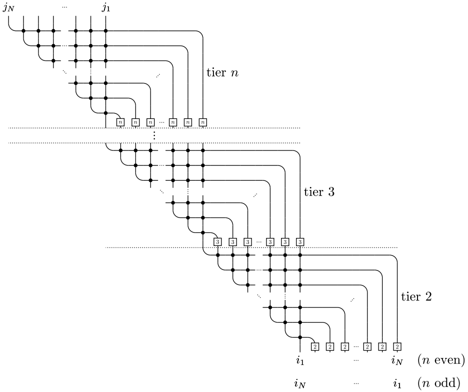

$$F_{\sigma(1)\ldots\sigma(N)} R^{\sigma} \quad = \quad F_{1\ldots N}$$

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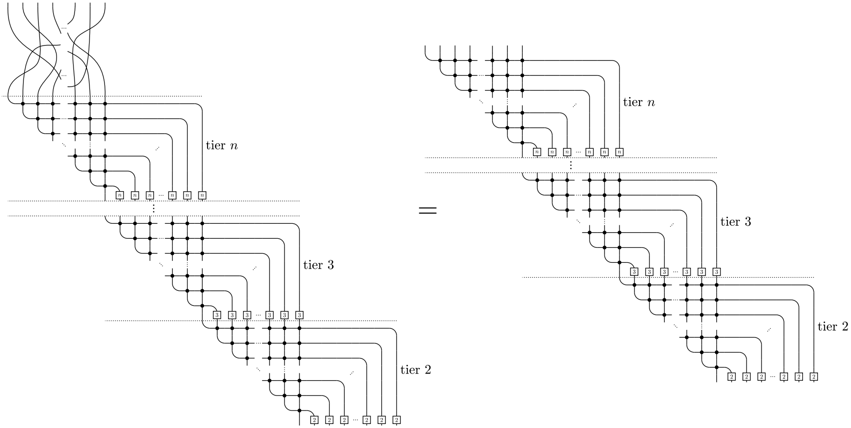

Lemmas

- tier-k Yang-Baxter equation

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- Lemma 1, for passing a tier-k R-matrix through a single tier-k partial F-matrix

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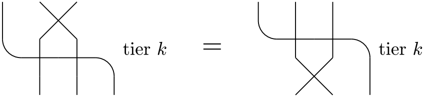
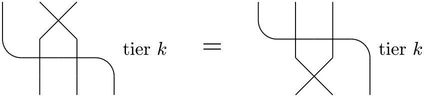
- Lemma 4, for passing a tier-k R-matrix through an F-matrix

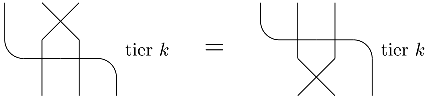
Yang-Baxter equation





Proof. Two cases:

- all colors $\leq k$
    - all R-matrices, so true by standard Yang-Baxter



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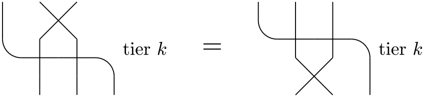
- one or more colors $> k$
    - at least two of the vertices are identity matrices and the statement holds



Proof. Two cases:

- all colors $\leq k$
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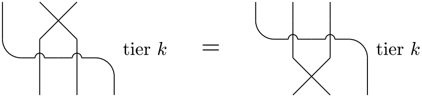
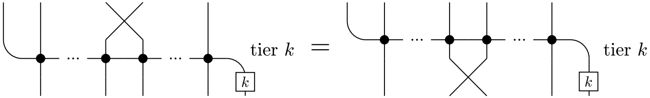
- one or more colors $> k$
    - at least two of the vertices are identity matrices and the statement holds
    - for example:

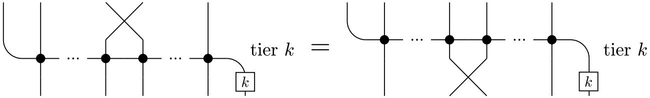


$$F^{k}_{1\ldots(i+1)i\ldots(N-1),N} R^{k}_{i(i+1)} = R^{k}_{i(i+1)} F^{k}_{1\ldots(N-1),N}$$

$$F^k_{1\dots(i+1)i\dots(N-1),N} R^k_{i(i+1)} = R^k_{i(i+1)} F^k_{1\dots(N-1),N}$$



$$F^{k}_{1\dots(i+1)i\dots(N-1),N}R^{k}_{i(i+1)} = R^{k}_{i(i+1)}F^{k}_{1\dots(N-1),N}$$

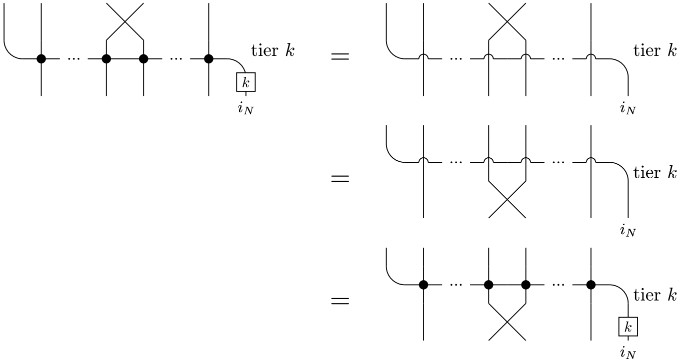


Proof. Two cases:

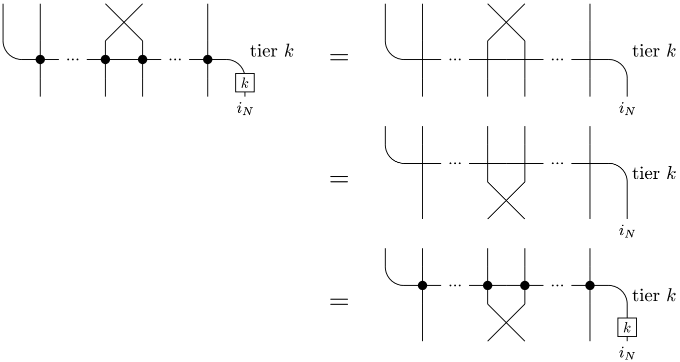
Case 1: $i_N = k$

Case 2: $i_N \neq k$

Case 1, $i_N = k$:



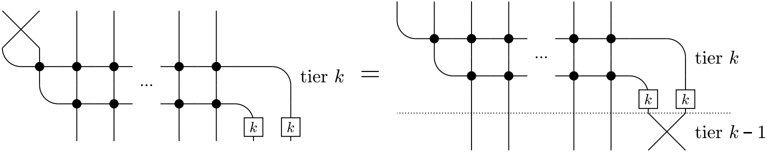
Case 2, $i_N \neq k$:



$$\begin{aligned} &F^{k}_{1\ldots(N-2),N}F^{k}_{1\ldots(N-2)N,(N-1)}R^{k}_{(N-1)N} \\ =\ &R^{k-1}_{N(N-1)}F^{k}_{1\ldots(N-2),(N-1)}F^{k}_{1\ldots(N-1),N} \end{aligned}$$

lemma 2

$$F^k_{1\ldots(N-2),N}F^k_{1\ldots(N-2)N,(N-1)}R^k_{(N-1)N}$$
$$= R^{k-1}_{N(N-1)}F^k_{1\ldots(N-2),(N-1)}F^k_{1\ldots(N-1),N}$$



$$F^k_{1\dots(N-2),N}F^k_{1\dots(N-2)N,(N-1)}R^k_{(N-1)N}$$
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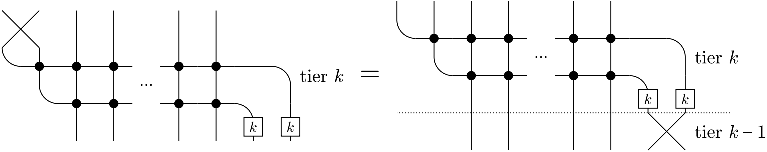
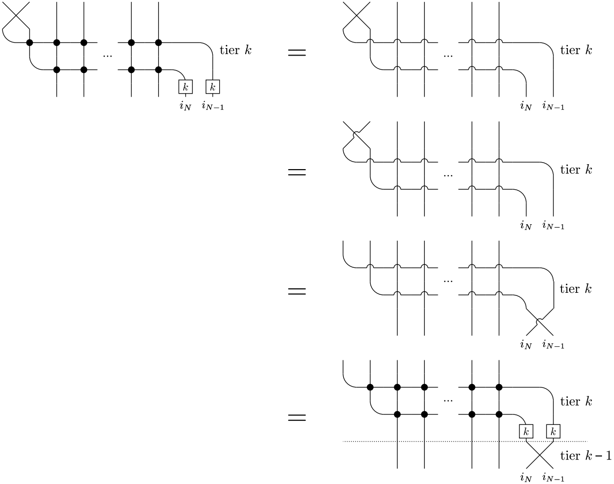


Proof. Four cases:

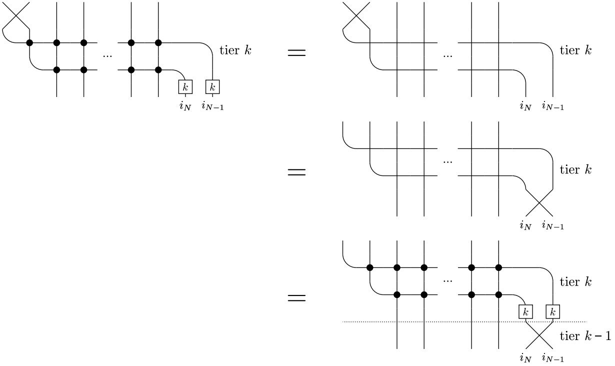
Case 1: $i_{N-1} = k, \quad i_N = k$ Case 2: $i_{N-1} \neq k, \quad i_N \neq k$

Case 3: $i_{N-1} \neq k, \quad i_N = k$ Case 4: $i_{N-1} = k, \quad i_N \neq k$

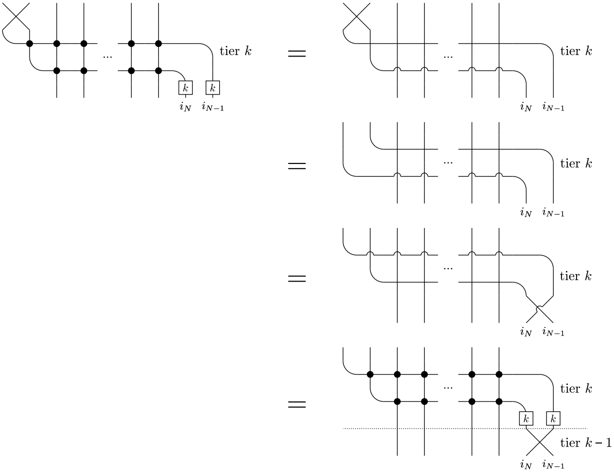
Case 1, $i_{(N-1)} = k, i_N = k$:



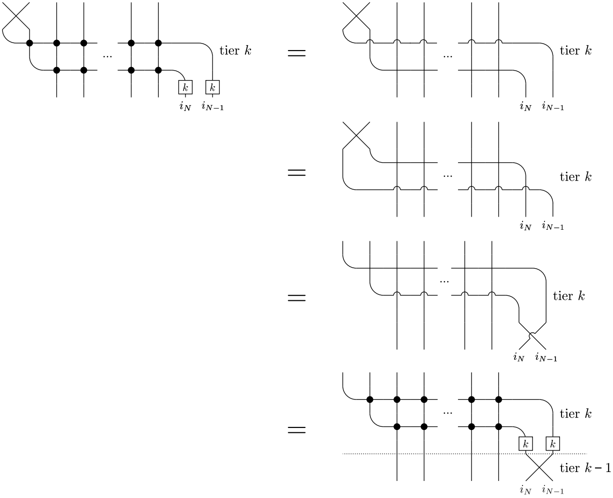
Case 2, $i_{(N-1)} \neq k, i_N \neq k$:



Case 3, $i_{N-1} \neq k, i_N = k$:



Case 4, $i_{N-1} = k, i_N \neq k$:



$$F^{k}_{1\ldots(i+1)i\ldots N}R^{k}_{i(i+1)} = R^{k-1}_{(i+1)i}F^{k}_{1\ldots N}$$

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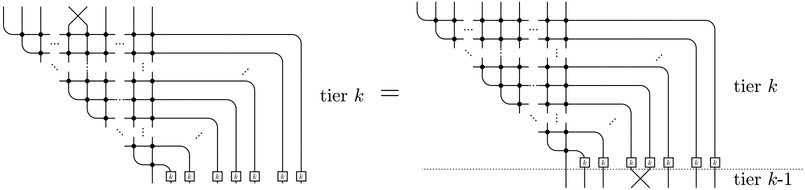
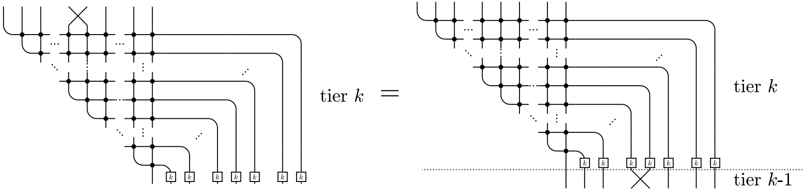


lemma 3

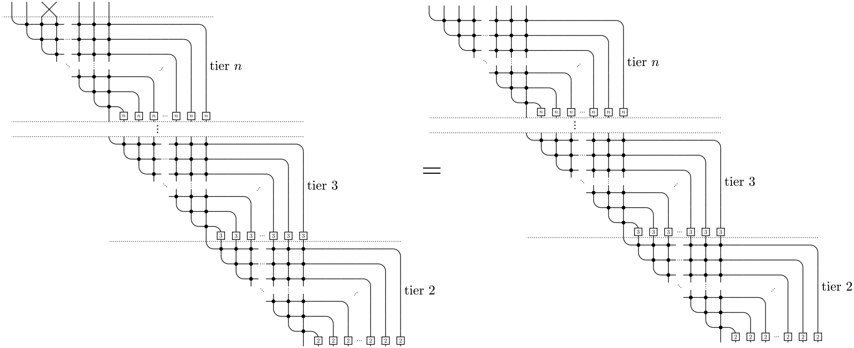
$$F^{k}_{1\dots(i+1)i\dots N}R^{k}_{i(i+1)} = R^{k-1}_{(i+1)i}F^{k}_{1\dots N}$$



Proof. Obvious!

$$
\begin{aligned}
F_{1\ldots(i+1)i\ldots N}R_{i(i+1)} &= \begin{cases} I_{(i+1)i}F_{1\ldots N}, & n \text{ even} \\ I_{i(i+1)}F_{1\ldots N}, & n \text{ odd} \end{cases} \\
&= F_{1\ldots N}
\end{aligned}
$$

$$F_{1\dots(i+1)i\dots N}R_{i(i+1)} = \begin{cases} I_{(i+1)i}F_{1\dots N}, & n \text{ even} \\ I_{i(i+1)}F_{1\dots N}, & n \text{ odd} \end{cases}$$
$$= F_{1\dots N}$$



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- $R_{12}^{n} = R_{12}$ and $R_{12}^{1} = I_{12}$

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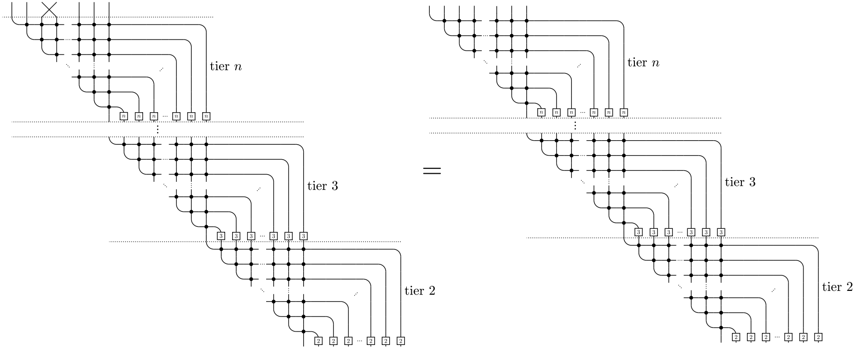
- then ...

lemma 4

Proof. Observe that

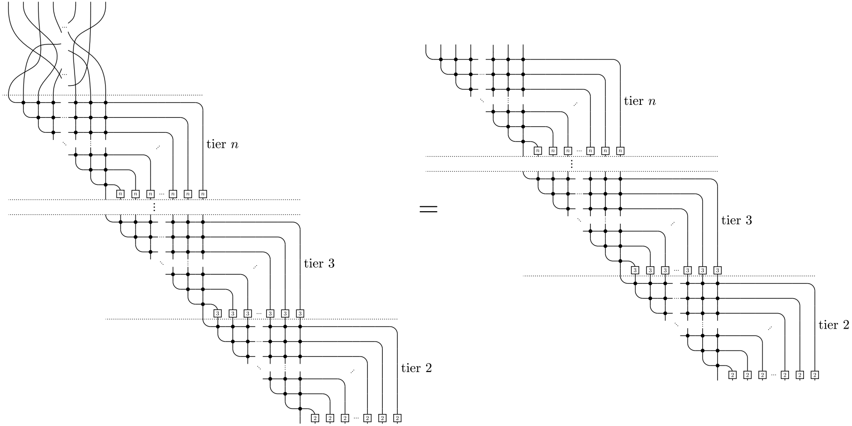
- $R^n_{12} = R_{12}$ and $R^1_{12} = I_{12}$
- then ... obvious!



$$F_{\sigma(1)\ldots\sigma(N)}R^{\sigma} \quad = \quad F_{1\ldots N}$$

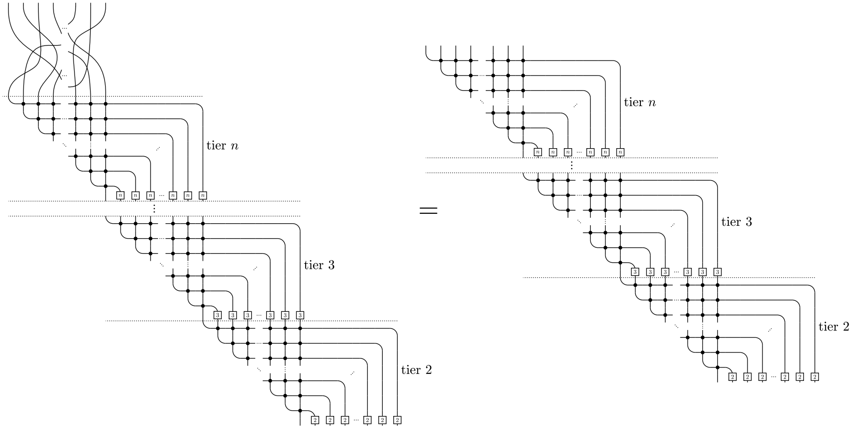
proof of theorem

$$F_{\sigma(1)\ldots\sigma(N)}R^{\sigma} \quad = \quad F_{1\ldots N}$$



proof of theorem

$$F_{\sigma(1)\dots\sigma(N)}R^{\sigma} \quad = \quad F_{1\dots N}$$



Proof. Obvious!

Discussion

Questions and comments.