Reduction formula of higher-spin XXZ form factors and the integrable XXZ model with spin-s quantum impurity [1]

Tetsuo Deguchi

Department of Physics, Ochanomizu University, Tokyo, Japan.

Main topics

- Reduction formula for the form factor of a local spin-s operator
- Multiple-integral representation of spin-s XXZ correlation functions
- Form factors for the spin-s quantum impurity in the XXZ spin chain

Many parts are in collaboration with **Jun Sato** and **Kohei Motegi**.

Recently, partially in collaboration with **Ryoko Yahagi** (quantum impurity) and **Takako Watanabe** (spin-1 form factors). We are also thankful to previous collaboration with **Chihiro Matsui**.

[1] Talk given in Dijon, Sep. 7, 2011

Main Reference of this talk

[D] T. D., arXiv:1105.4722. (reduction formula of spin-s form factors)

[DMo] T. D. and Kohei Motegi, in preparation.

[DS] T. D. and Jun Sato,
Quantum group $U_q(sl(2))$ symmetry and explicit evaluation of the one-point functions of the integrable spin-1 XXZ spin chain,
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[DM2] T. D. and Chihiro Matsui,
Correlation functions of the integrable higher-spin XXX and XXZ spin chains through the fusion method,
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[DM1] T. D. and Chihiro Matsui, Nucl. Phys. B **814** [FS] (2009) 405–438,

[DM4] T. D. and Chihiro Matsui, Nucl. Phys., B **851** (2011) pp. 238-243.
(Erratum to [DM1])

Contents

- **Part I**: Reduction of spin-s form factors through fusion method

  (1) Formula for expressing the spin-s operators with spin-1/2 ones (**revised verion to [DM1]**)

  (2): "Quantum Inverse Scattering Problem" for spin-s case

  We do not solve it for operators but for matrix elements (i.e., form factors).

- **Part II**: Physical application 1

  (1): Multiple-integral representation of arbitrary correlation functions for the integrable spin-s XXZ spin chain (**revised version of [DM2]**)

  (2): Numerical confirmation

- **Part III**: Physical application 2

  Form factors of the spin-s quantum impurity in the XXZ chain

- **Motivations**: **Super-integrable chiral Potts chain, Quantum Dynamics**

1 Integrable spin-s XXZ Hamiltonians

Spin-1/2 case:

The Hamiltonian of the XXZ spin chain under the periodic boundary conditions (P.B.C.) is given by

$$\mathcal{H}_{\rm XXZ} = \sum_{j=1}^{L} \left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right) \ . \tag{1}$$

Here σ_j^a $(a = X, Y, Z)$ are the Pauli matrices defined on the jth site and Δ denotes the anisotropy of the exchange coupling. The P.B.C. are given by $\sigma_{L+1}^a = \sigma_1^a$ for $a = X, Y, Z$.

Here we define

$$\Delta = (q + q^{-1})/2 \, , \qquad (q = \exp \eta).$$

$|\Delta| > 1$: massive regime

$|\Delta| < 1$: massless regime (CFT with $c = 1$).

The integrable spin-1 XXZ Hamiltonian

The spin-1 XXZ Hamiltonian under the P.B.C. is given by the following:

$$
\begin{aligned}
& \mathcal{H}_{\mathrm{spin-1XXZ}} \\
= \; & J \sum_{j=1}^{N_s} \Big\{ \vec{S}_j \cdot \vec{S}_{j+1} - (\vec{S}_j \cdot \vec{S}_{j+1})^2 - \frac{1}{2}(q - q^{-1})^2 [S_j^z S_{j+1}^z - (S_j^z S_{j+1}^z)^2 + 2(S_j^z)^2] \\
& -(q + q^{-1} - 2)[(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) S_j^z S_{j+1}^z + S_j^z S_{j+1}^z (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) \Big\} . \qquad (2)
\end{aligned}
$$

In the massless regime we assume $q = \exp i\zeta$ with $0 \leq \zeta < \pi/2s$. $(\eta = i\zeta)$
In the massive regime $q = \exp \zeta$ with $\zeta > 0$.

In the massless regime, the ground state of the integrable spin-s XXZ spin chain corresponds to the criticality described by the SU(2) WZW model with level $2s$.

We set $L = 2sN_s$. We often denote $2s$ by ℓ. (ℓ are integers.)

Spin-s XXZ Hamiltonian expressed with the q-Clebsch-Gordan coefficiants

$$\mathcal{H}_{\mathrm{XXZ}}^{(2s)} = \frac{d}{d\lambda} \log t_{12\cdots N_s}^{(2s,\, 2s)}(\lambda) \Big|_{\lambda=0,\, \xi_j=0} = \sum_{i=1}^{N_s} \frac{d}{du} \check{R}_{i,i+1}^{(2s,2s)}(u) \Big|_{u=0}$$

where $t_{12\cdots N_s}^{(2s,\, 2s)}(\lambda)$ denotes the transfer matrix of the integrable spin-s XXZ chain. Here, the elements of the R-matrix for $V(l_1) \otimes V(l_2)$ are given by (cf. [T.D. and K. Motegi])

$$\check{R}|l_1, a_1\rangle \otimes |l_2, a_2\rangle = \sum_{b_1, b_2} \check{R}_{a_1,a_2}^{b_1,b_2} |l_1, b_1\rangle \otimes |l_2, b_2\rangle,$$

$$\check{R}_{a_1,a_2}^{b_1,b_2} = \delta_{a_1+a_2, b_1+b_2} N(l_1, a_1) N(l_2, a_2) \sum_{j=0}^{\min(l_1,l_2)} N(l_1+l_2-2j, a_1+a_2)^{-1} \times \rho_{l_1+l_2-2j} \begin{bmatrix} l_2 & l_1 & l_1+l_2-2j \\ b_1 & b_2 & a_1+a_2 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_1+l_2-2j \\ a_1 & a_2 & a_1+a_2 \end{bmatrix}$$

2 Algebraic Bethe ansatz

We define the R-matrix and the monodromy matrix $T_{0,12\cdots L}(\lambda;\{w_j\})$ by

$$R_{12}(\lambda_1,\lambda_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]}$$

$$T_{0,12\cdots L}(\lambda;\{w_j\}) = R_{0L}(\lambda,w_L)R_{0L-1}(\lambda,w_{L-1})\cdots R_{02}(\lambda,w_2)R_{01}(\lambda,w_1)\,.$$

Here $u=\lambda_1-\lambda_2$, $b(u)=\sinh u/\sinh(u+\eta)$ and $c(u)=\sinh\eta/\sinh(u+\eta)$ with $q=\exp\eta$. The operator-valued matrix elements of give the "creation and annihilation operators"

$$T_{0,12\cdots L}(u;\{w_j\}) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_{[0]}\,.$$

The transfer matrix, $t(u)$, is given by the trace of the monodromy matrix with respect to the 0th space:

$$\begin{aligned} t(u;w_1,\ldots,w_L) &= \mathrm{tr}_0\left(T_{0,12\cdots L}(u;\{w_j\})\right) \\ &= A(u;\{w_j\})+D(u;\{w_j\})\,. \end{aligned} \tag{3}$$

3 Fusion method

First trick:

Applying the R-matrix R^+ in homogeneous grading to the fusion construction

Through a similarity transformation we transform R to R^+

$$R^+_{12}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^-(u) & 0 \\ 0 & c^+(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]} . \quad (4)$$

$$c^{\pm}(u) = e^{\pm u} \sinh\eta / \sinh(u+\eta) \quad b(u) = \sinh u / \sinh(u+\eta)$$

The R^+ gives the intertwiner of the affine quantum group $U_q(\widehat{sl_2})$ in the homogeneous grading of evaluation representations:

$$R^+_{12}(u)\,(\Delta(a))_{12} = (\tau \circ \Delta(a))_{12}\, R^+_{12}(u) \quad a \in U_q(sl_2)$$

where τ denotes the permutation operator: $\tau(a \otimes b) = b \otimes a$ for $a, b \in U_q(sl_2)$, and $(\Delta(a))_{12} = \pi_1 \otimes \pi_2\,(\Delta(a))$.

Gauge transformation

We define the gauge transformation $\chi_{12\cdots L}$ by

$$\chi_{12\cdots L} = \Phi_1(w_1)\Phi_2(w_2)\cdots\Phi_L(w_L)\,. \tag{5}$$

Here

$$\Phi(w) = \mathrm{diag}(1, \exp(w))$$

and w_j denote the inhomogeneity parameters of the spin-1/2 transfer matrix of the XXZ spin chain for $j = 1, 2, \ldots, L$.

Projection operators and the fusion constrution

We define permutation operator Π_{12} by

$$\Pi_{12} v_1 \otimes v_2 = v_2 \otimes v_1 \, , \tag{6}$$

and then define $\check{R}$ by

$$\check{R}^{+}_{12}(u) = \Pi_{12} R^{+}_{12} \tag{7}$$

We define spin-1 projection operator by

$$P^{(2)}_{12} = \check{R}^{+}_{12}(u = \eta) \tag{8}$$

We define $spin - \ell/2$ projection operator recursively as follows.

$$P^{(\ell)}_{12\cdots\ell} = P^{(\ell-1)}_{12\cdots\ell-1} \check{R}^{+}_{\ell-1,\ell}((\ell-1)\eta) P^{(\ell-1)}_{12\cdots\ell-1} \, , \tag{9}$$

We define monodromy matrix $T_0^{(1,\,2s)}(\lambda_0;\xi_1,\ldots,\xi_{N_s})$ acting on the tensor product $V^{(1)}(\lambda_0)\otimes\left(V^{(2s)}(\xi_1)\otimes\cdots\otimes V^{(2s)}(\xi_{N_s})\right)$ as follows.

$$T_0^{(1,\,2s)}(\lambda_0;\xi_1,\ldots,\xi_{N_s}) = P_{12\ldots L}^{(2s)}\,\cdot\,R_{0,12\cdots L}^{+}(\lambda_0;w_1^{(2s)},\ldots,w_L^{(2s)})\,\cdot\,P_{12\ldots L}^{(2s)}\,. \tag{10}$$

Here inhomogenous parameters w_j are given by complete $2s$-strings

$$w_{2s(p-1)+k}^{(2s)} = \xi_p-(k-1)\eta \quad (p=1,2,\ldots,N_s;k=1,\ldots,2s.)$$

More precisely, we shall put them as almost complete $2s$-strings

$$w_{2s(p-1)+k}^{(2s;\,\epsilon)} = \xi_p-(k-1)\eta+\epsilon r_k \quad (p=1,2,\ldots,N_s;k=1,\ldots,2s.)$$

We express the matrix elements of the monodromy matrix as follows.

$$T_{0,\,12\cdots N_s}^{(1,\,2s)}(\lambda;\{\xi_k\}_{N_s}) = \begin{pmatrix} A^{(2s)}(\lambda;\{\xi_k\}_{N_s}) & B^{(2s)}(\lambda;\{\xi_k\}_{N_s}) \\ C^{(2s)}(\lambda;\{\xi_k\}_{N_s}) & D^{(2s)}(\lambda;\{\xi_k\}_{N_s}) \end{pmatrix}. \tag{11}$$

$$A^{(2s)}(\lambda;\{\xi_k\}_{N_s}) = P_{12\cdots L}^{(2s)}\cdot A^{(1)}(\lambda;\{w_j^{(2s)}\}_L)\cdot P_{12\cdots L}^{(2s)}$$

Figure 1: Matrix element of the monodromy matrix $(T^{(\ell,\,2s)}_{\alpha,\beta})^{a_1,\ldots,a_{N_s}}_{b_1,\ldots,b_{N_s}}$.

Here quantum spaces $V_j^{(2s)}(\xi_j)$ are $(2s+1)$-dimensional (vertical lines),
Variables a_j and b_j take values $0, 1, \ldots, 2s$

while the auxiliary space $V_0^{(\ell)}(\lambda_0)$ is $(\ell+1)$-dimensional (horizontal line) . variables c_j take values $0, 1, \ldots, \ell$.

$$L = 2sN_s$$

Spin-1/2 chain of L-sites with inhomogeneous parameters $w_1, \ldots, w_L$,
while the spin-s chain of N_s sites with $\xi_1, \ldots, \xi_{N_s}$.

We now define $T_0^{(\ell,\,2s)}(\lambda_0; \xi_1, \ldots, \xi_{N_s})$ acting on the tensor product $V_0^{(\ell)}(\lambda_0) \otimes \left(V^{(2s)}(\xi_1) \otimes \cdots \otimes V^{(2s)}(\xi_{N_s})\right)$ as follows.

$$T^{(\ell,\,2s)}_{0,\,12\cdots N_s} = P^{(\ell)}_{a_1a_2\cdots a_\ell}\, T^{(1,\,2s)}_{a_1,\,12\cdots N_s}(\lambda_{a_1}) T^{(1,\,2s)}_{a_2,\,12\cdots N_s}(\lambda_{a_1}-\eta) \cdots T^{(1,\,2s)}_{a_\ell,\,12\cdots N_s}(\lambda_{a_1}-(\ell-1)\eta)\, P^{(\ell)}_{a_1a_2\cdots a_\ell}\,.$$

4 Qunatum groups

The quantum algebra $U_q(sl_2)$ is an associative algebra over $\mathbf{C}$ generated by $X^{\pm}, K^{\pm}$ with the following relations:

$$\begin{aligned} KK^{-1} &= KK^{-1} = 1\,, \quad KX^{\pm}K^{-1} = q^{\pm 2}X^{\pm}\,, \quad , \\ [X^+, X^-] &= \frac{K - K^{-1}}{q - q^{-1}}\,. \end{aligned} \tag{12}$$

The algebra $U_q(sl_2)$ is also a Hopf algebra over $\mathbf{C}$ with comultiplication

$$\begin{aligned} \Delta(X^+) &= X^+ \otimes 1 + K \otimes X^+\,, \quad \Delta(X^-) = X^- \otimes K^{-1} + 1 \otimes X^-\,, \\ \Delta(K) &= K \otimes K\,, \end{aligned} \tag{13}$$

and antipode: $S(K) = K^{-1}\,, S(X^+) = -K^{-1}X^+\,, S(X^-) = -X^-K$, and coproduct: $\epsilon(X^{\pm}) = 0$ and $\epsilon(K) = 1$.

$[n]_q = (q^n - q^{-n})/(q - q^{-1})$: the q-integer of an integer n.

$[n]_q!$: the q-factorial for an integer n.

$$[n]_q! = [n]_q \, [n-1]_q \, \cdots \, [1]_q \, . \tag{14}$$

For integers $m \geq n \geq 0$, the q-binomial coefficient is defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q! \, [n]_q!} \, . \tag{15}$$

We define $||\ell, 0\rangle$ for $n = 0, 1, \ldots, \ell$ by

$$||\ell, 0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_\ell \, . \tag{16}$$

Here $|\alpha\rangle_j$ for $\alpha = 0, 1$ denote the basis vectors of the spin-1/2 rep. We define $||\ell, n\rangle$ for $n \geq 1$ and evaluate them as follows .

$$\begin{aligned} ||\ell, n\rangle &= \left(\Delta^{(\ell-1)}(X^-)\right)^n ||\ell, 0\rangle \, \frac{1}{[n]_q!} \\ &= \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1}^- \cdots \sigma_{i_n}^- |0\rangle \, q^{i_1 + i_2 + \cdots + i_n - n\ell + n(n-1)/2} \, . \end{aligned} \tag{17}$$

We have conjugate vectors $\langle \ell, n||$ explicitly as folllows.

$$\langle \ell, n|| = \begin{bmatrix} \ell \\ n \end{bmatrix}_q^{-1} q^{n(\ell-n)} \sum_{1 \le i_1 < \cdots < i_n \le \ell} \langle 0|\sigma_{i_1}^{+} \cdots \sigma_{i_n}^{+} \, q^{i_1+\cdots+i_n - n\ell + n(n-1)/2} \, . \tag{18}$$

Here the normalization conditions: $\langle \ell, n|| \, ||\ell, n\rangle = 1$.

We can show

$$P^{(\ell)} = \sum_{n=0}^{\ell} ||\ell, n\rangle \langle \ell, n||. \tag{19}$$

Spin-$\ell/2$ elementary operators

We shall define spin-s elementary operators by

$$E^{i,\,j\,(\ell)} = ||\ell, i\rangle \langle \ell, j||$$

5 Reduction formula for the spin-s XXZ form factors

$||\ell, 0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_\ell$ we have the following:

$$
\begin{aligned}
||\ell, 0\rangle\langle \ell, 0|| &= |0\rangle_1 \otimes \cdots \otimes |0\rangle_\ell \ \langle 0|_1 \otimes \cdots \otimes \langle 0|_\ell \\
&= |0\rangle_1 \langle 0|_1 \otimes \cdots \otimes |0\rangle_\ell \langle 0|_\ell \\
&= e_1^{0,0} \cdots e_\ell^{0,0} \, . \qquad (20)
\end{aligned}
$$

We have

$$
|0\rangle_1\langle 0|_1 = (1,0)^T (1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e^{0,0} \, . \qquad (21)
$$

We consider spin-1/2 elementary operators $e^{\varepsilon', \varepsilon}$ for $\varepsilon', \varepsilon = 0, 1$, as follows.

$$
e^{0,1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad e^{1,0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad e^{1,1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
$$

For a given sequence, $a_1, a_2, \ldots, a_N$, we denote it by $(a_j)_N$, briefly; i.e., we have

$$(a_j)_N = (a_1, a_2, \ldots, a_N) \,. \tag{22}$$

For $(\varepsilon_\alpha^{'})_\ell$ and $(\varepsilon_\beta)_\ell$ consisting of only two values 0 or 1, we consider the following product:

$$\prod_{k=1}^{\ell} e_k^{\varepsilon_k^{'}, \varepsilon_k} = e_1^{\varepsilon_1^{'}, \varepsilon_1} \cdots e_\ell^{\varepsilon_\ell^{'}, \varepsilon_\ell} \,. \tag{23}$$

We define a set $\boldsymbol{\alpha}^-$ by the set of integers k satisfying $\varepsilon_k^{'} = 1$ for $1 \le k \le \ell$ and a set $\boldsymbol{\alpha}^+$ by the set of integers k satisfying $\varepsilon_k = 0$ for $1 \le k \le \ell$, respectively:

$$\boldsymbol{\alpha}^-(\{\varepsilon_\alpha^{'}\}) = \{\alpha ;\, \varepsilon_\alpha^{'} = 1 \, (1 \le \alpha \le \ell)\} \,, \quad \boldsymbol{\alpha}^+(\{\varepsilon_\beta\}) = \{\beta ;\, \varepsilon_\beta = 0 \, (1 \le \beta \le \ell)\} \,. \tag{24}$$

Let us denote by Σ_ℓ the set of integers $1, 2, \ldots, \ell$;

$$\Sigma_\ell = \{1, 2, \ldots, \ell\}.$$

In terms of sets $\boldsymbol{\alpha}^\pm$ we express the product of elementary operators given by (23) as

$$\prod_{k=1}^{\ell} e_k^{\varepsilon_k^{'}, \varepsilon_k} = \prod_{a \in \boldsymbol{\alpha}^-} \sigma_a^- ||\ell, 0\rangle \, \langle \ell, 0|| \prod_{b \in \Sigma_\ell \setminus \boldsymbol{\alpha}^+} \sigma_b^+ \,. \tag{25}$$

We express the elements of $\boldsymbol{\alpha}^-$ as $a(k)$ for $k = 1, 2, \ldots, i$, and those of $\Sigma_\ell \setminus \boldsymbol{\alpha}^+$ as $b(k)$ for $k = 1, 2, \ldots, j$, respectively.

$$\boldsymbol{\alpha}^- = \{a(1), a(2), \ldots, a(i)\}\,, \quad \Sigma_\ell \setminus \boldsymbol{\alpha}^+ = \{b(1), b(2), \ldots, b(j)\}\,. \tag{26}$$

Suppose that we have a sequence $(\varepsilon_\alpha^{'})_\ell$ such that $\varepsilon_\alpha^{'} = 0$ or 1 for all integers α with $1 \le \alpha \le \ell$ and the number of integers α satisfying $\varepsilon_\alpha^{'} = 1$ $(1 \le \alpha \le \ell)$ is given by i. Then, we denote $\varepsilon_\alpha^{'}$ by $\varepsilon_\alpha^{'}(i)$ for each integer α and the sequence $(\varepsilon_\alpha^{'})_\ell$ by $(\varepsilon_\alpha^{'}(i))_\ell$.

Sequences $(\varepsilon_\alpha^{'}(i))_\ell$ and $(\varepsilon_\beta(j))_\ell$ are related to integers $a(1) < a(2) < \cdots < a(i)$ and $b(1) < b(2) < \cdots < b(j)$, respectively, by

$$e_1^{\varepsilon_1^{'}(i),\,\varepsilon_1(j)} \cdots e_\ell^{\varepsilon_\ell^{'}(i),\,\varepsilon_\ell(j)} = e_{a(1)}^{1,\,0} \cdots e_{a(i)}^{1,\,0}\, e_1^{0,\,0} \cdots e_\ell^{0,\,0}\, e_{b(1)}^{0,\,1} \cdots e_{b(j)}^{0,\,1}\,, \tag{27}$$

$$\prod_{k=1}^{\ell} e_k^{\varepsilon_k^{'}(i),\,\varepsilon_k(j)} = \prod_{a \in \boldsymbol{\alpha}^-} \sigma_a^- \, ||\ell, 0\rangle \, \langle \ell, 0|| \prod_{b \in \Sigma_\ell \setminus \boldsymbol{\alpha}^+} \sigma_b^+\,. \tag{28}$$

We define spin-$\ell/2$ elementary operators associated with grading w by

$$E^{i,j\,(\ell\, w)} = ||\ell, i\rangle\langle \ell, j||$$

We have

$$||\ell, i\rangle\langle \ell, j|| = \sum_{(\varepsilon_{\alpha}^{'}(i))_{\ell}} \sum_{(\varepsilon_{\beta}(j))_{\ell}} g_{ij}(\varepsilon_{\alpha}^{'}(i),\, \varepsilon_{\beta}(j)) e_{1}^{\varepsilon_{1}^{'}(i),\, \varepsilon_{1}(j)} \cdots e_{\ell}^{\varepsilon_{\ell}^{'}(i),\, \varepsilon_{\ell}(j)}\,. \tag{29}$$

Here the sum is taken over all two sequences $(\varepsilon_{\alpha}^{'}(i))_{\ell}$ and $(\varepsilon_{\beta}(j))_{\ell}$.

Lemma 1. *Let* $\boldsymbol{\alpha}^{-}$ *be a set of distinct integers* $\{a(1), \ldots, a(i)\}$ *satisfying* $1 \le a(1) < \ldots < a(i) \le \ell$, *we have the following:*

$$\langle \ell, i|| \sigma_{a(1)}^{-} \cdots \sigma_{a(i)}^{-} ||\ell, 0\rangle \, q^{-(a(1)+\cdots+a(i))+i} = \begin{bmatrix} \ell \\ i \end{bmatrix}_{q}^{-1} q^{-i(i-1)/2}\,, \tag{30}$$

which is independent of the set $\boldsymbol{\alpha}^{-} = \{a(1), a(2), \ldots, a(i)\}$.

Proposition 1. *For every pair of integers i and j with $1 \le i, j \le \ell$ the spin-$\ell/2$ elementary operator associated with grading w, $E_1^{i,j\,(\ell\, w)}$, is decomposed into a sum of products of the spin-1/2 elementary operators as follows.*

$$
\begin{aligned}
E_1^{i,j\,(\ell\, w)} \ &= \ \begin{bmatrix} \ell \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ j \end{bmatrix}_q^{-1} q^{i(i-1)/2 - j(j-1)/2} e^{-(i-j)\xi_1} \\
& \quad \times P_{12\ldots\ell}^{(\ell)} \sum_{(\varepsilon_\beta(j))_\ell} \chi_{12\cdots\ell} \, e_1^{\varepsilon_1^{'}(i), \varepsilon_1(j)} \cdots e_\ell^{\varepsilon_\ell^{'}(i), \varepsilon_\ell(j)} \chi_{12\cdots\ell}^{-1} \, . \qquad (31)
\end{aligned}
$$

Here, we fix a sequence $(\varepsilon_\alpha^{'}(i))_\ell$. Furthermore, the expression (31) does not depend on the order of $\varepsilon_\alpha^{'}(i)$ s with respect to αs.

Quantum inverse-scattering problem

Let us recall the formula of the quantum inverse-scattering problem (QISP) for the spin-1/2 XXZ spin chain (Kitanine et al, 1999)

$$x_n = \prod_{k=1}^{n-1} \left(A^{(1\,w)} + D^{(1\,w)} \right)(w_k) \cdot \mathrm{tr}_0 \left(x_0 T^{(1\,w)}_{0,\,12\cdots L}(w_n) \right) \cdot \prod_{k=1}^{n} \left(A^{(1\,w)} + D^{(1\,w)} \right)^{-1} (w_k)\,. \tag{32}$$

Here we assume that inhomogeneity parameters w_j are given by generic values so that the transfer matrices $\left(A^{(1\,w)} + D^{(1\,w)}\right)(w_k)$ are regular for $k = 1, 2, \ldots, n$.

Making use of the QISP formula (32) we have the following expressions for $b = 1, 2, \ldots, N_s$:

$$\begin{aligned} e^{\varepsilon_1^{'}, \varepsilon_1}_{\ell(b-1)+1} \cdots e^{\varepsilon_\ell^{'}, \varepsilon_\ell}_{\ell(b-1)+\ell} &= \prod_{k=1}^{\ell(b-1)} \left(A^{(1\,w)}(w_k) + D^{(1\,w)}(w_k) \right) \\ &\times T^{(1\,w)}_{\varepsilon_1,\,\varepsilon_1^{'}}(w_{\ell(b-1)+1}) \cdots T^{(1\,w)}_{\varepsilon_\ell,\,\varepsilon_\ell^{'}}(w_{\ell(b-1)+\ell}) \prod_{k=1}^{\ell b} \left(A^{(1\,w)}(w_k) + D^{(1\,w)}(w_k) \right)^{-1}\,. \end{aligned} \tag{33}$$

Here we have denoted by $T_{\alpha,\beta}(\lambda)$ the (α, β) element of the monodromy matrix $T(\lambda)$.

“Quantum inverse-scattering problem” for the spin-$\ell/2$ operators

$$\widehat{E}_1^{i\,j\,(\ell\,w)} = \widehat{N}_{i,\,j}^{(\ell)}\, e^{-(i-j)\Lambda_1 \delta(w,p)} \cdot P_{1\cdots\ell}^{(\ell)} \times$$

$$\times \; \chi_{12\cdots\ell} \sum_{(\varepsilon_\beta(j))_\ell} T^{(1\,w)}_{\varepsilon_1(j),\,\varepsilon_1^{'}(i)}(w_1)\cdots T^{(1\,w)}_{\varepsilon_\ell(j),\,\varepsilon_\ell^{'}(i)}(w_\ell) \prod_{k=1}^{\ell} \Big(A^{(1\,w)}(w_k) + D^{(1\,w)}(w_k) \Big)^{-1} \chi^{-1}_{12\cdots\ell} \, .$$

Here, we fix a sequence $(\varepsilon_\alpha^{'}(i))_\ell$.

Problem: Non-regularity of the transfer matrix

Putting $\lambda = w_1^{(2)} = \xi_1$ we have

$$A_{12}^{(2\,+;\,0)}(\xi_1) + D_{12}^{(2\,+;\,0)}(\xi_1) = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{[2]_q} & \frac{q^{-1}}{[2]_q} & 0 \\ 0 & \frac{q}{[2]_q} & \frac{1}{[2]_q} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)_{[1,2]} . \qquad (34)$$

We thus show that the transfer matrix is non-regular at $\lambda = w_1^{(2)} = \xi_1$:

$$\det \Big(A_{12}^{(2\,+;\,0)}(\xi_1) + D_{12}^{(2\,+;\,0)}(\xi_1) \Big) = 0. \qquad (35)$$

A "solution" to the spin-s QISP through continuity assumption of solutions of the Bethe-ansatz equations

Let us now assume that the Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ approach the Bethe roots $\{\lambda_\beta\}_M$ continuously in the limit of sending ϵ to 0. It follows that each entry of the Bethe-ansatz eigenstate of the Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ is continuous with respect to ϵ. For a set of arbitrary parameters $\{\mu_k\}_N$ we therefore have

$$
\begin{aligned}
&\langle 0| \prod_{\alpha=1}^{N} C^{(\ell\, p;\, 0)}(\mu_a) \, \cdot \, e_1^{\varepsilon_1^{'}, \varepsilon_1} \cdots e_\ell^{\varepsilon_\ell^{'}, \varepsilon_\ell} \, \cdot \, \prod_{\beta=1}^{M} B^{(\ell\, p;\, 0)}(\lambda_\beta)|0\rangle \\
&= \lim_{\epsilon \to 0} \langle 0| \prod_{\alpha=1}^{N} C^{(\ell\, p;\, \epsilon)}(\mu_a) \, \cdot \, e_1^{\varepsilon_1^{'}, \varepsilon_1} \cdots e_\ell^{\varepsilon_\ell^{'}, \varepsilon_\ell} \, \cdot \, \prod_{\beta=1}^{M} B^{(\ell\, p;\, \epsilon)}(\lambda_\beta(\epsilon))|0\rangle \, . \qquad (36)
\end{aligned}
$$

Solution to the spin-s QISP for the matrix elements (form factors)

We have the following expressions for $b = 1, 2, \ldots, N_s$:

$$e_{\ell(b-1)+1}^{\varepsilon_1^{'},\varepsilon_1} \cdots e_{\ell(b-1)+\ell}^{\varepsilon_\ell^{'},\varepsilon_\ell} = \prod_{k=1}^{\ell(b-1)} \left(A^{(\ell\, w;\,\epsilon)}(w_k^{(\ell;\,\epsilon)}) + D^{(\ell\, w;\,\epsilon)}(w_k^{(\ell;\,\epsilon)}) \right)$$

$$\times T_{\varepsilon_1,\varepsilon_1^{'}}^{(\ell\, w;\,\epsilon)}(w_{\ell(b-1)+1}^{(\ell;\,\epsilon)}) \cdots T_{\varepsilon_\ell,\varepsilon_\ell^{'}}^{(\ell\, w;\,\epsilon)}(w_{\ell(b-1)+\ell}^{(\ell;\,\epsilon)}) \prod_{k=1}^{\ell b} \left(A^{(\ell\, w;\,\epsilon)}(w_k^{(\ell;\,\epsilon)}) + D^{(\ell\, w;\,\epsilon)}(w_k^{(\ell;\,\epsilon)}) \right)^{-1} .$$

For instance in the case of $b = 1$, we have

$$\langle 0| \prod_{\alpha=1}^{N} C^{(\ell\, p;\,\epsilon)}(\mu_a) \cdot e_1^{\varepsilon_1^{'},\,\varepsilon_1} \cdots e_\ell^{\varepsilon_\ell^{'},\,\varepsilon_\ell} \cdot \prod_{\beta=1}^{M} B^{(\ell\, p;\,\epsilon)}(\lambda_\beta(\epsilon))|0\rangle$$

$$= \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\})\, \langle 0| \prod_{\alpha=1}^{N} C^{(\ell\, p;\,\epsilon)}(\mu_a) \cdot T_{\varepsilon_1,\,\varepsilon_1^{'}}^{(\ell\, p;\,\epsilon)}(w_1^{(\ell;\,\epsilon)}) \cdots T_{\varepsilon_\ell,\,\varepsilon_\ell^{'}}^{(\ell\, p;\,\epsilon)}(w_\ell^{(\ell;\epsilon)}) \cdot$$

$$\times \prod_{\beta=1}^{M} B^{(\ell\, p;\,\epsilon)}(\lambda_\beta(\epsilon))|0\rangle . \tag{37}$$

Proposition 2. *Let* $\{\mu_k\}_N$ *be a set of arbitrary parameters and* $\{\lambda_\alpha\}_M$ *a solution of the spin-*$\ell/2$ *Bethe-ansatz equations. We denote by* $\{\lambda_\alpha(\epsilon)\}_M$ *a solution of the Bethe-ansatz equations for the spin-1/2 XXZ chain whose inhomogeneity parameters* w_j *are given by the* N_s *pieces of the almost complete* ℓ*-strings:* $w_j = w_j^{(\ell;\,\epsilon)}$ *for* $1 \le j \le L$. *We assume that the set* $\{\lambda_\alpha(\epsilon)\}_M$ *approaches* $\{\lambda_\alpha\}_M$ *continuously when we send* ϵ *to zero. For the Bethe states* $\langle \{\mu_k\}_N |$ *and* $|\{\lambda_\alpha\}_M\rangle$*, which are off-shell and on-shell, respectively, we evaluate the matrix elements of a given product of elementary operators* $e_1^{\varepsilon_1^{'},\varepsilon_1} \cdots e_\ell^{\varepsilon_\ell^{'},\varepsilon_\ell}$ *as follows.*

$$
\begin{aligned}
\langle 0| \prod_{\alpha=1}^{N} C^{(\ell\, p;\,0)}(\mu_a)\, e_1^{\varepsilon_1^{'},\varepsilon_1} \cdots e_\ell^{\varepsilon_\ell^{'},\varepsilon_\ell} \prod_{\beta=1}^{M} B^{(\ell\, p;\,0)}(\lambda_\beta)|0\rangle &= \phi_\ell(\{\lambda_\beta\};\{w_j^{(\ell)}\}) \\
\times \lim_{\epsilon\to 0} \langle 0| \prod_{\alpha=1}^{N} C^{(\ell\, p;\,\epsilon)}(\mu_a)\, T^{(\ell\, p;\,\epsilon)}_{\varepsilon_1,\,\varepsilon_1^{'}}(w_1^{(\ell;\,\epsilon)}) \cdots T^{(\ell\, p;\,\epsilon)}_{\varepsilon_\ell,\,\varepsilon_\ell^{'}}(w_\ell^{(\ell;\,\epsilon)}) & \prod_{\beta=1}^{M} B^{(\ell\, p;\,\epsilon)}(\lambda_\beta(\epsilon))|0\rangle\,,
\end{aligned}
\tag{38}
$$

where $\phi_m(\{\lambda_\beta\})$ *has been defined by* $\phi_m(\{\lambda_\beta\};\{w_j\}) = \prod_{j=1}^{m} \prod_{\alpha=1}^{M} b(\lambda_\alpha - w_j)$ *with* $b(u) = \sinh(u)/\sinh(u+\eta)$.

General spin-$\ell/2$ elementary operators

In the spin-$\ell/2$ representation constructed in the ℓth tensor product space $(V^{(1)})^{\otimes \ell}$, we define the general spin-s elementary operators associated with principal grading, $\widehat{E}^{i,\,j\,(\ell\,p)}$, by

$$\widehat{E}^{i,\,j\,(\ell\,p)} = ||\ell, i\rangle\, \langle \ell, j||\, \frac{g(j)}{g(i)}\,, \quad \text{for } i, j = 0, 1, \ldots, \ell. \tag{39}$$

Then, through the spin-$\ell/2$ gauge transformation we define the general spin-s elementary operators associated with homogeneous grading by

$$\widehat{E}^{i,\,j\,(\ell\,+)} = \chi_{12\ldots N_s}^{(\ell)}\, \widehat{E}^{i,\,j\,(\ell\,p)} \, \left(\chi_{12\ldots N_s}^{(\ell)}\right)^{-1} \,. \tag{40}$$

We explicitly have

$$\widehat{E}^{i,\,j\,(\ell\,+)} = ||\ell, i\rangle\, \langle \ell, j||\, \frac{g(j)}{g(i)}\, e^{(i-j)(\xi-(\ell-1)\eta/2)}, \quad \text{for } i, j = 0, 1, \ldots, \ell. \tag{41}$$

Here we recall that the quantity $\xi - (\ell - 1)\eta/2$ corresponds to the string center of the ℓ-string: $\xi, \xi - \eta, \ldots, \xi - (\ell - 1)\eta$.

We define the general spin-$\ell/2$ elementary operators associated with principal grading acting in the tenor product space $V_1^{(\ell)} \otimes \cdots V_{N_s}^{(\ell)}$ by

$$\widehat{E}_k^{i,\, j\, (\ell\, p)} = (I^{(\ell)})^{\otimes (k-1)} \otimes \widehat{E}^{i,\, j\, (\ell\, p)} \otimes (I^{(\ell)})^{\otimes (N_s - k)} \, , \quad \text{for } i, j = 0, 1, \ldots, \ell. \tag{42}$$

Similarly we define that of homogeneous grading, $\widehat{E}_k^{i,\, j\, (\ell,\, +)}$ for $i, j = 0, 1, \ldots, \ell$.

Let us introduce the normalization factor $\widehat{N}_{i,\, j}^{(\ell)}$ by $\widehat{N}_{i,\, j}^{(\ell)} = N_{i,\, j}^{(\ell)} g(i)/g(j)$. We have

$$\widehat{N}_{i,\, j}^{(\ell)} = \frac{g(j)}{g(i)} \frac{F(\ell, i)}{F(\ell, j)} \, q^{i(\ell - i)/2 - j(\ell - j)/2}. \tag{43}$$

We define $\delta(w, p)$ for gradings $\pm$ and p by

$$\delta(w, p) = \begin{cases} 1 & \text{if } w = p, \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

With factor $\widehat{N}_{i,\, j}^{(\ell)}$ and the string center: $\Lambda_1 = \xi_1 - (\ell - 1)\eta/2$, from Proposition 1, we have

$$\widehat{E}_1^{i,\, j\, (\ell\, w)} = \widehat{N}_{i,\, j}^{(\ell)} \, e^{-(i-j)\Lambda_1 \, \delta(w,p)} \, P_{12\ldots\ell}^{(\ell)} \sum_{(\varepsilon_\beta(j))_\ell} \chi_{12\cdots\ell} \, e_1^{\varepsilon_1^{'}(i),\, \varepsilon_1(j)} \cdots e_\ell^{\varepsilon_\ell^{'}(i),\, \varepsilon_\ell(j)} \, \chi_{12\cdots\ell}^{-1} \, . \tag{45}$$

Here, we recall that sequence $(\varepsilon_\alpha^{'}(i))_\ell$ is fixed.

Proposition 3. *Let us take integers i_k and j_k satisfying $1 \leq i_k, j_k \leq \ell$ for $k = 1, 2, \ldots, m$. We set $\sum_k i_k - \sum_k j_k = N - M$. Let $\{\mu_k\}_N$ be a set of arbitrary N parameters. If the set of the Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ approaches the set of the Bethe roots $\{\lambda_\beta\}_M$ continuously at $\epsilon = 0$, we have the following:*

$$
\begin{aligned}
& \langle 0| \prod_{\alpha=1}^{N} C^{(\ell\, w)}(\mu_a) \cdot \prod_{k} \widehat{E}_k^{i_k,\, j_k\, (\ell\, w)} \cdot \prod_{\beta=1}^{M} B^{(\ell\, w)}(\lambda_\beta)|0\rangle \\
& = \left(\prod_{k=1}^{m} \widehat{N}_{i_k,\, j_k}^{(\ell_k)} \right) \cdot e^{\sigma(w)(\sum_{k=1}^{N} \mu_k - \sum_{\gamma=1}^{M} \lambda_\gamma)} \, \phi_\ell(\{\lambda_\beta\}; \{w_j^{(\ell)}\}) \sum_{(\varepsilon_\beta^{[1]}(j_1))_\ell} \cdots \sum_{(\varepsilon_\beta^{[m]}(j_m))_\ell} \\
& \times \lim_{\epsilon \to 0} \langle 0| \prod_{\alpha=1}^{N} C^{(\ell\, p;\, \epsilon)}(\mu_a) \prod_{k=1}^{m} \left(T^{(\ell\, p;\, \epsilon)}_{\varepsilon_1^{[k]}(j_k),\, \varepsilon_1^{[k]\prime}(i_k)}(w_1^{(\ell;\, \epsilon)}) \cdots T^{(\ell\, p;\, \epsilon)}_{\varepsilon_\ell^{[k]}(j_k),\, \varepsilon_\ell^{[k]\prime}(j_k)}(w_\ell^{(\ell;\, \epsilon)}) \right) \\
& \quad \times \prod_{\beta=1}^{M} B^{(\ell\, p;\, \epsilon)}(\lambda_\beta(\epsilon))|0\rangle \, . \qquad (46)
\end{aligned}
$$

Here we have chosen sequences $\varepsilon_\alpha^{[k]\prime}(j_k)$ for each integer k of $1 \leq k \leq m$.

Here we consider a product of the general spin-$\ell/2$ elementary operators,

$$\hat{E}_1^{i_1, j_1 \, (\ell \, w)} \cdots \hat{E}_m^{i_m, j_m \, (\ell \, w)}.$$

We also recall variables $\varepsilon_\alpha^{[k] \, \prime}(i_k)$ and $\varepsilon_\beta^{[k]}(j_k)$ which take only two values 0 or 1 for $k = 1, 2, \ldots, m$ and $\alpha, \beta = 0, 1, \ldots, \ell$. We have the following:

For the mth product of elementary operators, we introduce the sets of variables $\varepsilon_\alpha^{[k] \, \prime}$s and $\varepsilon_\beta^{[k]}$s $(1 \le k \le m)$ such that the number of $\varepsilon_\alpha^{[k] \, \prime} = 1$ with $1 \le a \le 2s$ is given by i_k and the number of $\varepsilon_\beta^{[k]} = 1$ with $1 \le b \le 2s$ by j_k, respectively. Here, the variables $\varepsilon_\alpha^{[k] \, \prime}$ and $\varepsilon_\beta^{[k]}$ take only two values 0 or 1. We then express them by integers $\varepsilon_j^{'}$s and ε_js for $j = 1, 2, \ldots, 2sm$ as follows:

$$\begin{aligned} \varepsilon_{2s(k-1)+\alpha}^{'} &= \varepsilon_\alpha^{[k] \, \prime} \quad \text{for} \quad \alpha = 1, 2, \ldots, 2s; k = 1, 2, \ldots, m, \\ \varepsilon_{2s(k-1)+\beta} &= \varepsilon_\beta^{[k]} \quad \text{for} \quad \beta = 1, 2, \ldots, 2s; k = 1, 2, \ldots, m. \end{aligned} \tag{47}$$

6 Multiple-integral representation for the spin-s XXZ CFs

The fundamental conjecture of the spin-s ground state

The spin-s ground state $|\psi_g^{(2s)}\rangle$ is given by $N_s/2$ sets of $2s$-strings for $0 \le \zeta < \pi/2s$.

$$\lambda_a^{(\alpha)} = \mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)} \, , \quad \text{for } a = 1, 2, \ldots, N_s/2 \text{ and } \alpha = 1, 2, \ldots, 2s.$$

Deviaions are given by $\epsilon_a^{(\alpha)} = \sqrt{-1}\delta_a^{(\alpha)}$ where $\delta_a^{(\alpha)}$ are real and decreasing w.r.t. α, and $|\delta_a^{(\alpha)}| > |\delta_a^{(\alpha+1)}|$ for $\alpha < s$, $|\delta_a^{(\alpha)}| < |\delta_a^{(\alpha+1)}|$ for $\alpha > s$.

Numerical solutions are given by Jun Sato.

The ground state should correspond to the criticality of the SU(2) WZW model with level $k = 2s$ $(c = 3s/(s+1))$.

For the homogeneous chain where $\xi_p = 0$ for $p = 1, 2, \ldots, N_s$, we denote the density of string centers by $\rho(\lambda)$.

$$\rho(\lambda) = \frac{1}{2\zeta \cosh(\pi\lambda/\zeta)} \, . \tag{48}$$

Multiple integral representations of the correlation function for an arbitrary product of elementary operators

We define a spin-s correlation function by

$$\widehat{F}^{(2sw)}(\{\epsilon_j, \epsilon_j^{'}\}) = \langle \psi_g^{(2sw)} | \prod_{i=1}^{m} \widehat{E}_i^{m_i\, , \, n_i\, (2sw)} | \psi_g^{(2sw)} \rangle / \langle \psi_g^{(2sw)} | \psi_g^{(2sw)} \rangle \qquad (49)$$

For the mth product of elementary operators, we introduce the sets of variables $\varepsilon_\alpha^{[k]\, '}$s and $\varepsilon_\beta^{[k]}$s $(1 \le k \le m)$ such that the number of $\varepsilon_\alpha^{[k]\, '} = 1$ with $1 \le a \le 2s$ is given by i_k and the number of $\varepsilon_\beta^{[k]} = 1$ with $1 \le b \le 2s$ by j_k, respectively. Here, the variables $\varepsilon_\alpha^{[k]\, '}$ and $\varepsilon_\beta^{[k]}$ take only two values 0 or 1. We then express them by integers $\varepsilon_j^{'}$s and ε_js for $j = 1, 2, \ldots, 2sm$ as follows:

$$\begin{aligned} \varepsilon_{2s(k-1)+\alpha}^{'} &= \varepsilon_\alpha^{[k]\, '} \quad \text{for} \quad \alpha = 1, 2, \ldots, 2s; k = 1, 2, \ldots, m, \\ \varepsilon_{2s(k-1)+\beta} &= \varepsilon_\beta^{[k]} \quad \text{for} \quad \beta = 1, 2, \ldots, 2s; k = 1, 2, \ldots, m. \end{aligned} \qquad (50)$$

Let us define $\boldsymbol{\alpha}^-$ and $\boldsymbol{\alpha}^+$ by

$$\boldsymbol{\alpha}^- = \{ j; \epsilon_j = 0 \} \, , \quad \boldsymbol{\alpha}^+ = \{ j; \epsilon_j^{'} = 1 \} \, . \qquad (51)$$

For sets $\boldsymbol{\alpha}^-$ and $\boldsymbol{\alpha}^+$ we define $\tilde{\lambda}_j$ for $j \in \boldsymbol{\alpha}^-$ and $\tilde{\lambda}_j^{'}$ for $j \in \boldsymbol{\alpha}^+$, respectively, by the following relation:

$$(\tilde{\lambda}^{'}_{j^{'}_{max}}, \ldots, \tilde{\lambda}^{'}_{j^{'}_{min}}, \tilde{\lambda}_{j_{min}}, \ldots, \tilde{\lambda}_{j_{max}}) = (\lambda_1, \ldots, \lambda_{2sm}) \,. \tag{52}$$

We have

$$\begin{aligned}
& \widehat{F}^{(2sw)}(\{\epsilon_j, \epsilon_j^{'}\}) = \widehat{C}(\{i_k, j_k\}) \times \\
= & \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\lambda_1 \\
& \cdots \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\lambda_{s^{'}} \\
& \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\lambda_{s^{'}+1} \\
& \cdots \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\lambda_m \\
& \quad \times \sum_{\boldsymbol{\alpha}^+(\{\epsilon_j\})} Q(\{\epsilon_j, \epsilon_j^{'}\}; \lambda_1, \ldots, \lambda_{2sm}) \, \mathrm{det} S(\lambda_1, \ldots, \lambda_{2sm})
\end{aligned} \tag{53}$$

Here factor Q is given by

$$
\begin{aligned}
& Q(\{\epsilon_j, \epsilon_j^{'}\}) \\
= \ & (-1)^{\alpha_+} \frac{\prod_{j \in \boldsymbol{\alpha}^-} \left(\prod_{k=1}^{j-1} \varphi(\tilde{\lambda}_j - w_k^{(2s)} + \eta) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}_j - w_k^{(2s)}) \right)}{\prod_{1 \le k < \ell \le 2sm} \varphi(\lambda_\ell - \lambda_k + \eta + \epsilon_{\ell,k})} \\
& \times \frac{\prod_{j \in \boldsymbol{\alpha}^+} \left(\prod_{k=1}^{j-1} \varphi(\tilde{\lambda}_j^{\prime} - w_k^{(2s)} - \eta) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}_j^{\prime} - w_k^{(2s)}) \right)}{\prod_{1 \le k < \ell \le 2sm} \varphi(w_k^{(2s)} - w_\ell^{(2s)})}
\end{aligned}
\tag{54}
$$

The matrix elements of S are given by

$$
S_{j,k} = \rho(\lambda_j - w_k^{(2s)} + \eta/2)\, \delta(\alpha(\lambda_j), \beta(k)) \,, \quad \text{for} \quad j, k = 1, 2, \ldots, 2sm \,. \tag{55}
$$

Here $\delta(\alpha, \beta)$ denotes the Kronecker delta and $\alpha(\lambda_j)$ are given by a if $\lambda_j = \mu_j - (a - 1/2)\eta$ $(1 \le a \le 2s)$, where μ_j correspond to centers of complete $2s$-strings.

In the denominator, we have set $\epsilon_{k,l}$ associated with λ_k and λ_l as follows.

$$
\epsilon_{k,l} = \begin{cases} i\epsilon & \text{for} \quad \mathcal{I}m(\lambda_k - \lambda_l) > 0 \\ -i\epsilon & \text{for} \quad \mathcal{I}m(\lambda_k - \lambda_l) < 0. \end{cases} \tag{56}
$$

The coefficient $\widehat{C}^{(2s)}(\{i_k, j_k\})$ is given by

$$\begin{aligned} \widehat{C}^{(2s)}(\{i_k, j_k\}) &= \prod_{k=1}^{m} \widehat{N}^{(\ell)}_{i_k, j_k} \\ &= \prod_{k=1}^{m} \left(\frac{g(j_k)}{g(i_k)} \, \frac{F(2s, i_k)}{F(2s, j_k)} q^{i_k(2s-i_k)/2 - j_k(2s-j_k)/2} \right) . \end{aligned} \tag{57}$$

If we put $g(2s, j) = \sqrt{F(2s, j)}$ for $j = 0, 1, \ldots, 2s$ into (57), we have

$$\widehat{C}^{(2s)}(\{i_k, j_k\}) = \prod_{k=1}^{m} \sqrt{ \left[\begin{array}{c} 2s \\ i_k \end{array} \right]_q \left[\begin{array}{c} 2s \\ j_k \end{array} \right]_q^{-1} } . \tag{58}$$

We may take any $\boldsymbol{\alpha}^{-}(\{\varepsilon_j^{'}\})$ corresponding to $\varepsilon_\alpha^{[k]\,'}$s for $k = 1, 2, \ldots, m$, as far as the number of $\varepsilon_\alpha^{[k]\,'} = 1$ with $1 \le \alpha \le 2s$ is given by i_k for each k.

7 Evaluating the integrals for the spin-1 one-point function

Evaluating the multiple integrals explicitly, we have obtained all the one-point function for the integrable spin-1 XXZ chain as

$$
\begin{aligned}
E^{2,\,2\,(2\,p)}\rangle = \langle E^{0,\,0\,(2\,p)}\rangle &= \frac{\zeta - \sin\zeta\cos\zeta}{2\zeta\sin^2\zeta}, \\
\langle E^{1,\,1\,(2\,p)}\rangle &= \frac{\cos\zeta(\sin\zeta - \zeta\cos\zeta)}{\zeta\sin^2\zeta}.
\end{aligned} \tag{59}
$$

In particular, via evaluation of the multiple integrals, we have confirmed the uniaxial symmetry relation:

$$
\langle E^{22}\rangle = \langle E^{00}\rangle\,. \tag{60}
$$

Through the direct evaluation of the multiple integrals we confirm the identity: $\langle E^{22}\rangle + \langle E^{11}\rangle + \langle E^{00}\rangle = 1$.

Furthermore, we have confirmed the relations among the correlation functions:

$$
\langle E^{1,\,1\,(2\,p)}\rangle = 2\,\langle e_1^{0,0} e_2^{1,1}\rangle = 2\,\langle e_1^{1,1} e_2^{0,0}\rangle = 2\,\langle e_1^{0,1} e_2^{1,0}\rangle = 2\,\langle e_1^{1,0} e_2^{0,1}\rangle\,. \tag{61}
$$

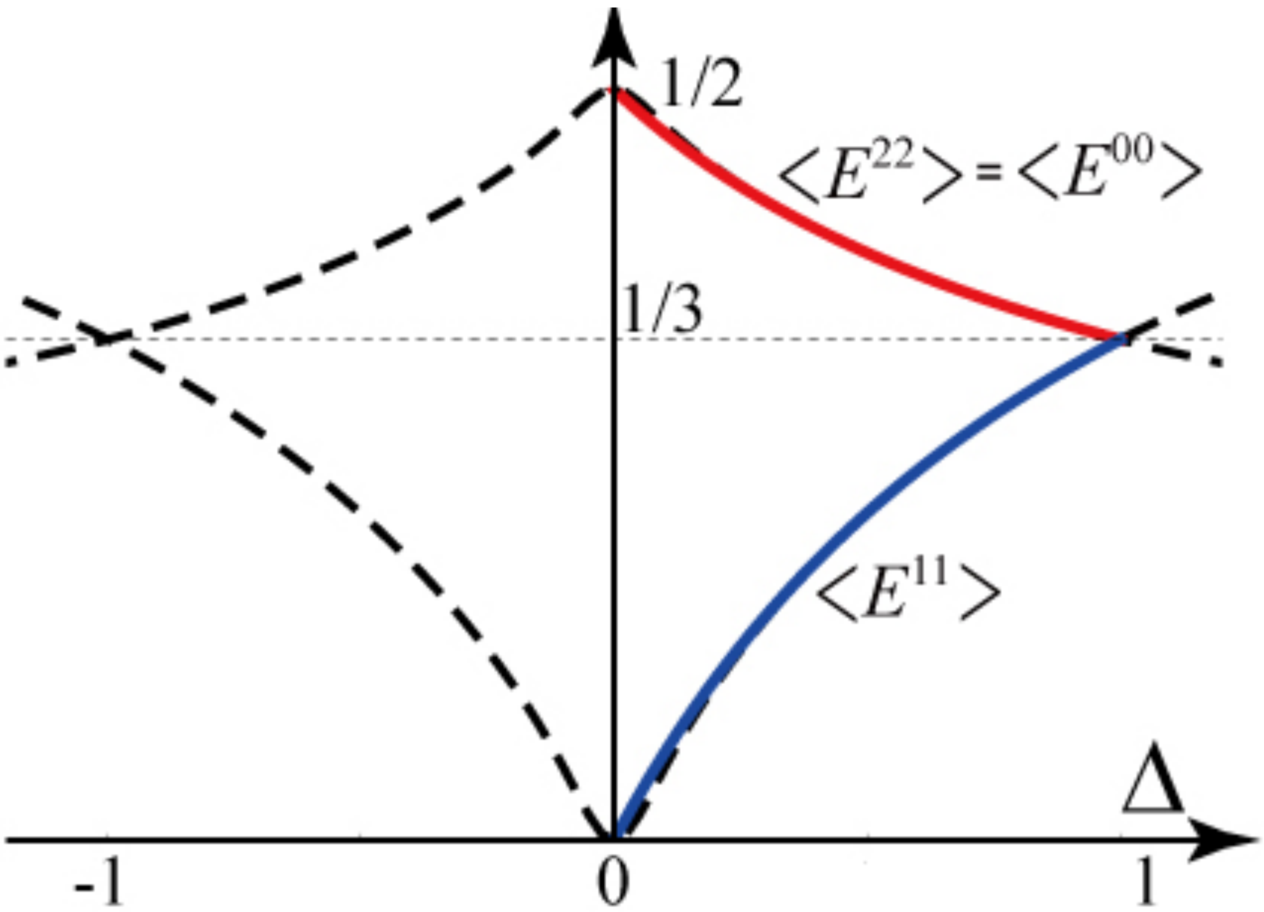

Figure 2: Comparison with the exact numerical diagonalization. The red and blue lines represent analytical results obtained by the multiple integrals for $\langle E^{22} \rangle = \langle E^{00} \rangle$ and $\langle E^{11} \rangle$, respectively. The black dotted lines represent those obtained by exact diagonalization with the system size $N_s = 8$. (Due to Jun Sato)

8 Spin-s quantum impurity: Form factors of the impurity

Let us consider the fusion transfer matrix whose quantum state is given by

$$V^{(2s)} \otimes V^{(1)} \otimes \cdots \otimes V^{(1)}$$

Here the spin-s site corresponds to the quantum impurity.

(1) N. Andrei and H. Johannesson, Phys. Lett. A (1984) pp. 108-112;

(2) P. Schlottmann, Nucl. Phys. B 552 (1999) pp. 727-747.

Proposition 4. *Let i_1 and j_1 be integers with $1 \le i_1, j_1 \le 2s$. We set $i_1 - j_1 = N - M$. Let $\{\mu_k\}_N$ be arbitrary. For a set of Bethe roots $\{\lambda_\beta(\epsilon)\}_M$ which approaches $\{\lambda_\beta\}_M$ continuously at $\epsilon = 0$ we have*

$$\langle 0| \prod_{\alpha=1}^{N} C^{(\mathrm{mx}\, w)}(\mu_a) \cdot \widehat{E}_1^{i_1, j_1\, (2s\, w)} \cdot \prod_{\beta=1}^{M} B^{(\mathrm{mx}\, w)}(\lambda_\beta)|0\rangle = \widehat{N}_{i_1, j_1}^{(2s)} \, e^{\sigma(w)(\sum_k \mu_k - \sum_\gamma \lambda_\gamma)}$$

$$\times \sum_{(\varepsilon_\beta(j_1))_\ell} \lim_{\epsilon \to 0} \langle 0| \prod_{\alpha=1}^{N} C^{(\mathrm{mx}\, p;\, \epsilon)}(\mu_a) \, T^{(\mathrm{mx}\, p;\, \epsilon)}_{\varepsilon_1(j_1), \varepsilon_1'(i_1)}(w_1^{(\mathrm{mx};\, \epsilon)}) \cdots T^{(\mathrm{mx}\, p;\, \epsilon)}_{\varepsilon_{2s}(j_1), \varepsilon_{2s}'(j_{2s})}(w_{2s}^{(\mathrm{mx};\, \epsilon)})$$

$$\times \prod_{\beta=1}^{M} B^{(\mathrm{mx}\, p;\, \epsilon)}(\lambda_\beta(\epsilon))|0\rangle \ \phi_{2s}(\{\lambda_\beta\}; \{w_j^{(\mathrm{mx})}\}) \, . \qquad (\mathrm{mx} = 2s \otimes 1 \otimes \cdots \otimes 1) \qquad (62)$$

9 Quantum Dynamics: 1D bosons interacting with δ-function potentials

The Lieb-Liniger Hamiltoninan is given by

$$\mathcal{H}_{LL} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j,k=1}^{N} c\,\delta(x_j - x_k)\,.$$

We introduce field operators for the 1D bosons, $\psi(x)$, $\psi(x)^\dagger$ satisfying the commutation relations:

$$[\psi(x), \psi^\dagger(y)] = \delta(x-y)$$

In the second quantized form, we have for $\mathcal{H}_{LL}$

$$\mathcal{H} = \int_0^L \{\partial_x\psi^\dagger(x)\partial_x\psi(x) + c\,\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)\}dx$$

The field operators satisfy the nonlinear Schrödinger equation:

$$i\partial_t\psi = -\partial_x^2\psi + 2c\psi^\dagger\psi^\dagger\psi$$

Density-density dynamical correlation function

The density operator is defined by

$$\rho(x,t) = \psi^\dagger(x,t)\psi(x,t)$$

The density-density dynamical correlation function, $G_2(x,t)$, is defined by

$$\begin{aligned}\langle \rho(x,t)\rho(0,0)\rangle &= \langle g|\rho(x,t)\rho(0,0)|g\rangle/\langle g|g\rangle \\ &= \sum_\mu \langle g|\rho(x,t)|\mu\rangle\langle\mu|\rho(0,0)|g\rangle/\langle g|g\rangle\langle\mu|\mu\rangle \\ &= \sum_\mu e^{i(E_g-E_\mu)t-(P_g-P_\mu)x} W(\mu,\lambda_g) \\ W(\mu,\lambda_g) &= |F(\mu,\lambda_g)|^2/\langle g|g\rangle\langle\mu|\mu\rangle\end{aligned}$$

Here, $|g\rangle$ denote the ground state, and λ_g the set of rapidities for the ground state. The form factor $F(\mu,\lambda_g)$ can be evaluated through Slavnov's formula. It is expressed in terms of a determinant.

Summary

- **Part I**: Reduction of spin-s form factors through fusion method

  (1) Formula for expressing the spin-s operators with spin-1/2 ones (**revised verion**)

  (2): "Quantum Inverse Scattering Problem" for spin-s case

  We do not solve it for operators but for matrix elements (i.e., form factors).

- **Part II**: Physical application 1

  (1): Multiple-integral representation of arbitrary correlation functions for the integrable spin-s XXZ spin chain (massless) (**revised version**)

  (2): Numerical confirmation

- **Part III**: Physical application 2

  Form factors of the spin-s quantum impurity in the XXZ chain

- **Motivations**: **Super-integrable chiral Potts chain, Quantum Dynamics**