FERMIONIC STRUCTURE IN THE SINE-GORDON MODEL : DIJON TALK PART III FORM FACTORS

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1. What I am going to tell.

Fedya and Michio talked about a set of Fermion Creation Operators which create Local Fields in the XXZ model and its CFT and SG limits. Let us call them the BJMS Fermions.

I am going to tell about another set of fermion operators creating the SG Form Factors, which characterize local fields in the SG model. Actually such fermions were introduced by Babelon-Bernard-Smirnov; so we call them the BBS Fermions. Our goal is to show that

$$\text{BBS Fermions} = \text{BJMS Fermions}$$

2. Form factors: what they are

The $2n$-particle form factors of a local field $\mathcal{O}_\alpha$ in the SG model are a set of $(\mathbb{C}^2)^{\otimes 2n}$-valued analytic functions in the $2n$ variables $\beta_1, \dots, \beta_{2n}$:

$$f_{\mathcal{O}_\alpha}(\beta_1, \dots, \beta_{2n})$$

Recall that the DG model is given by the action

$$\mathcal{A}^{\mathrm{sG}} = \int \Big[\frac{1}{16\pi}(\partial_\mu \boldsymbol{\varphi}(x))^2 + \frac{\boldsymbol{\mu}^2}{\sin\pi\beta^2} 2\cos(\beta\boldsymbol{\varphi}(x))\Big] d^2x.$$

where the parameter ν is given by $\nu = 1 - \beta^2$. Here $\mathcal{O}_\alpha$ is a descendant of the primary field $\Phi_\alpha(z, \bar{z}) = e^{i\alpha\frac{\nu}{2\sqrt{1-\nu}}\varphi(z,\bar{z})}$.

The form factors are characterized by the three axioms:

Symmetry axiom

$$S_{j,j+1}(\beta_j - \beta_{j+1}) f_{\mathcal{O}_\alpha}(\cdots, \beta_j, \beta_{j+1}, \cdots) = f_{\mathcal{O}_\alpha}(\cdots, \beta_{j+1}, \beta_j, \cdots)$$

Riemann-Hilbert problem axiom

$$f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-1}, \beta_{2n} + 2\pi i) = e^{-\frac{\pi i\nu}{1-\nu}\alpha\sigma^3_{2n}} f_{\mathcal{O}_\alpha}(\beta_{2n}, \beta_1, \cdots, \cdots, \beta_{2n-1})\,.$$

Residue axiom

$$\begin{aligned}&2\pi i\ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n}) = \\ &\Big(1 - e^{-\frac{\pi i\nu}{1-\nu}\alpha\sigma^3_{2n}} S_{2n-1,1}(\beta_{2n-1} - \beta_1) \cdots S_{2n-1,2n-2}(\beta_{2n-1} - \beta_{2n-2})\Big) \\ &\times f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-2}) \otimes s_{2n-1,2n}\,,\end{aligned}$$

Note that the shift of the parameter $\alpha \to \alpha + 2m\frac{1-\nu}{\nu}$ makes no difference in the axioms. This is a Hint because the BJMS fermions decsribe all $\alpha + 2m\frac{1-\nu}{\nu}$ together.

You might ask:

What are the form factors?

How are they related to the local operator $\mathcal{O}_\alpha$?

I will answer to this question when we discuss the XXZ lattice.

3. Form factors: what we do

Our problem is formulated as follows.

The axioms do not tell which descendant $\mathcal{O}_{\alpha+2m\frac{1-\nu}{\nu}}$ we are talking about. We want to know of which local fields we are constructing form factors. Since BJMS fermions do tell which local fields they create, we want to know which form factors we create by the BJMS fermions.

Our strategy is as follows.

We give integral representations of form factors. Integrals are parametrized by a tower of functions $L^{(n)}(S_1,\dots,S_n)$ called deformed Q-forms, where $S_1,\dots,S_n$ are integration variables. For $m=0$ and $\mathcal{O}_\alpha=\Phi_\alpha$ we have a simple formula for $L^{(n)}$, which we denote by

$$\begin{aligned} M_0^{(n)}(S_1,\cdots,S_n) &= \langle\Phi_\alpha\rangle\cdot S\wedge S^3\wedge\cdots\wedge S^{2n-1} \\ &= \langle\Phi_\alpha\rangle\cdot\prod_{j=1}^{n}S_j\prod_{1\le i<j\le n}(S_j^2-S_i^2). \end{aligned}$$

BBS fermions act on the space of towers, and create general $L^{(n)}$ out of $M_0^{(n)}$.

By definition, form factors $f_{\mathcal{O}_\alpha}(\beta_1,\dots,\beta_{2n})$ are the matrix elements of a local field $\mathcal{O}_\alpha$ between the vacuum state and the excited state of $2n$ solitons with the rapidity variables $\beta_1,\dots,\beta_{2n}$. If the descendant $\mathcal{O}_\alpha$ is created by BJMS fermions, its matrix elements are explicitly given by the famous ω in the form of determinants. Therefore, if we identify the towers created by the BBS fermions with the determinants for the local fields created by the BJMS fermions, we do the job.

Let us do.

4. BJMS fermions with solitons

Recall BJMS fermions on the XXZ lattice.

Problem is to compute the ratio of the matrix elements between the left and right eigenvectors of the twisted 松原 transfer matrix.

$$Z^{\kappa}_{\mathbf{n}}\{q^{2\alpha S(0)}\mathcal{O}\} = \frac{\langle N_-, \kappa+\alpha|\mathrm{Tr}_S\left(T_{S,\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\mathcal{O}\right)|N_+,\kappa\rangle}{\langle N_-, \kappa+\alpha|\mathrm{Tr}_S\left(T_{S,\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\right)|N_+,\kappa\rangle}.$$

Here $\mathbf{n}$ is the length of the 松原 direction and κ is the twist parameter. $\mathcal{O}$ is a local oeprator. We take the right eigenvector to be the vacuum $|N_+,\kappa\rangle = |0,\kappa\rangle$, the left a $2n$ particle excited state, where the twist parameter is $\kappa+\alpha$. In the limit $\mathbf{n}\to\infty$ the twist parameter is irrelevant. We have $|0,\kappa\rangle \to |\mathrm{vac}\rangle$. In this limit the eigenvalues of the twisted transfer matrix are parametrized by a set of $2n$ real numbers $\beta_1,\dots,\beta_{2n}$ and are $\binom{2n}{n}$-fold degenerate. Let us denote such an eigenvector by $\langle\beta_1,\dots,\beta_{2n};\ell^{(n)}|$. Here $\ell^{(n)}$ is choosing one of the $\binom{2n}{n}$ eigenvectors. The functional reduces to the ratio of the matrix elements

$$Z^{\kappa}_{\mathbf{n}}\{q^{2\alpha S(0)}\mathcal{O}\} \to \frac{\langle\beta_1,\dots,\beta_{2n};\ell^{(n)}|q^{2\alpha S(0)}\mathcal{O}|\mathrm{vac}\rangle}{\langle\beta_1,\dots,\beta_{2n};\ell^{(n)}|q^{2\alpha S(0)}|\mathrm{vac}\rangle}$$

The fermionic determinant formula reads

$$\det\left(\omega(\zeta_i,\xi_j;\ell^{(n)})\right)_{i,j=1,\cdots k} = \frac{\langle\beta_1,\cdots,\beta_{2n};\ell^{(n)}|\mathbf{b}^*(\zeta_1)\cdots\mathbf{b}^*(\zeta_k)\mathbf{c}^*(\xi_k)\cdots\mathbf{c}^*(\xi_1)q^{2\alpha S(0)}|\mathrm{vac}\rangle}{\langle\beta_1,\cdots,\beta_{2n};\ell^{(n)}|q^{2\alpha S(0)}|\mathrm{vac}\rangle}.$$

Before proceeding to the computation of $\omega(\zeta_i,\xi_j;\ell^{(n)})$, we discuss form factors and BBS fermions.

5. Form Factors and Bethe vectors

Form factors are expressed as

$$f_{\mathcal{O}_\alpha}(\beta_1,\cdots,\beta_{2n}) = Z(\beta_1,\cdots,\beta_{2n})$$

$$\times \sum_{I^-\sqcup I^+=\{1,\dots,2n\}} w^{\epsilon_1,\cdots,\epsilon_{2n}}(\beta_1,\cdots,\beta_{2n}) \frac{e^{\frac{\nu}{2(1-\nu)}\left(\sum_{p\in I^-}\beta_p-\sum_{p\in I^+}\beta_p+n\pi i\right)}}{\prod_{p\in I^-,q\in I^+}\sinh\frac{\nu}{1-\nu}(\beta_p-\beta_q)} \cdot \mathcal{F}_{\mathcal{O}_\alpha,n}(\beta_{I^-}|\beta_{I^+})$$

Here $Z(\beta_1,\cdots,\beta_{2n})$ is a normalization factor, and the most essential part $\mathcal{F}_{\mathcal{O}_\alpha,n}(\beta_{I^-}|\beta_{I^+})$ will be given in the next section.

The vectors given by

$$w^{\epsilon_1,\cdots,\epsilon_{2n}}(\beta_1,\cdots,\beta_{2n}) = \prod_{j:\ \epsilon_j=+} C(\mathfrak{b}_j)|\downarrow\rangle$$

form a basis of $(\mathbb{C}^2)^{\otimes 2n}$. Here $\mathfrak{b}_j = e^{\frac{2\nu}{1-\nu}\beta_j}$ and

$$\begin{pmatrix} A(\mathfrak{t}) & B(\mathfrak{t}) \\ C(\mathfrak{t}) & D(\mathfrak{t}) \end{pmatrix}_a = \widetilde{S}_{a,2n}(\mathfrak{t}/\mathfrak{b}_{2n})\cdots\widetilde{S}_{a,1}(\mathfrak{t}/\mathfrak{b}_1)\,.$$

are the monodromy operators. The sum is over the partition of $\{1,\dots,2n\}$ into two parts such that $\{j;\epsilon_j=-\}=I^-$, $\{j;\epsilon_j=+\}=I^+$ and $\sharp(I^\pm)=n$. Symmetry axiom is satisfied.

Now we give an answer to the question: What are the form factors?

To the vector $\langle\beta_1,\dots,\beta_{2n};\ell^{(n)}|$ obtained in the limit $\mathbf{n}\to\infty$ there corresponds a Bethe vector in $(\mathbb{C}^2)^{\otimes 2n}$. The correspondence reads as

$$\langle\beta_1,\cdots,\beta_{2n};\ell^{(n)}| \leftrightarrow \langle\uparrow| \prod_{j=1}^{n} C(\mathfrak{u}_j)$$

so that we have the equality

$$\langle\beta_1,\cdots,\beta_{2n};\ell^{(n)}|\mathcal{O}_\alpha|\mathrm{vac}\rangle = \langle\uparrow|\prod_{j=1}^{n} C(\mathfrak{u}_j)\cdot f(\beta_1,\cdots,\beta_{2n})\,.$$

6. Integrals

Now we give the integral representation to $\mathfrak{F}_{\mathcal{O}_{\alpha},n}(\beta_{I^-}|\beta_{I^+})$. Goal is the RH axiom. We do a fine tuning of polynomials which appear in the integrand.

We use

$$B_j = e^{\beta_j}, \ \mathfrak{b}_j = e^{\frac{2\nu}{1-\nu}\beta_j}, \ S = e^{\sigma}, \ \mathfrak{s} = e^{\frac{2\nu}{1-\nu}\sigma}.$$

The latter half are used as integration variables. S has a period $2\pi i$ in σ and $\mathfrak{s}$ $\frac{1-\nu}{\nu}\pi i$.

Let $\chi(\sigma)$ be a solution to the difference equations in each period.

$$\chi(\sigma + 2\pi i)p(\mathfrak{s}\mathfrak{q}^4) = \chi(\sigma)p(\mathfrak{s}\mathfrak{q}^2),$$
$$\chi(\sigma + \tfrac{1-\nu}{\nu}\pi i)P(SQ) = \chi(\sigma)P(-S)\,,$$

where

$$P(S) = \prod_{j=1}^{2n}(S - B_j)\,, \quad p(\mathfrak{s}) = \prod_{j=1}^{2n}(\mathfrak{s} - \mathfrak{b}_j)\,.$$

We define a pairing between Laurent polynomials $\ell(\mathfrak{s})$ and $L(S)$ by

$$(\mathfrak{s}^m, S^k)_\alpha = I_{\alpha+2m+\frac{1-\nu}{\nu}k}(\beta_1, \ldots, \beta_{2n}),$$
$$I_\alpha(\beta_1, \ldots, \beta_{2n}) = \int_{\mathbb{R}-i0} \chi(\sigma|\beta_1, \ldots, \beta_{2n})e^{\frac{\nu}{1-\nu}\alpha\sigma}d\sigma$$

Note that changing S^k to S^{k+1} is equivalent to the shift $\alpha \to \alpha + \frac{1-\nu}{\nu}$.

Now we define anti-symmetric n form in s.

$$\ell^{(n)}_{I^-\sqcup I^+}(\mathfrak{s}_1, \cdots, \mathfrak{s}_n) = \left(\ell_{I^-\sqcup I^+,0} \wedge \cdots \wedge \ell_{I^-\sqcup I^+,n-1}\right)(\mathfrak{s}_1, \cdots, \mathfrak{s}_n)\,,$$
$$\ell_{I^-\sqcup I^+,i}(\mathfrak{s}) = a^{-1}\Big\{p_{I^-}(\mathfrak{s})\left(p_{I^+,i}(\mathfrak{s}) - p_{I^+,i}(\mathfrak{s}\mathfrak{q}^2)\right)$$
$$+ \mathfrak{q}^{2(i-n)}p_{I^+}(\mathfrak{s}\mathfrak{q}^2)\left(p_{I^-,i}(\mathfrak{s}) - a^2 p_{I^-,i}(\mathfrak{s}\mathfrak{q}^2)\right)\Big\},$$
$$p_{I^-}(\mathfrak{s}) = \prod_{j\in I^-}(\mathfrak{s} - \mathfrak{b}_j), \quad p_{I^+}(\mathfrak{s}) = \prod_{j\in I^+}(\mathfrak{s} - \mathfrak{b}_j)\,,$$

Here we used

$$p_{I^\pm,i}(\mathfrak{s}) = \left[\mathfrak{s}^{i-n}p_{I^\pm}(\mathfrak{s})\right]_{\geq}\,,$$

where $[\ ]_{\geq}$ signifies the polynomial part.

The Riemann-Hilbert problem axiom (12.2) is satisfied if we set

$$\mathfrak{F}_{\mathcal{O}_{\alpha},n}(\beta_{I^-}|\beta_{I^+}) = e^{\frac{\nu}{2(1-\nu)}(1-\alpha)(\sum\limits_{j=1}^{2n}\beta_j - \pi i n)} \cdot (\ell^{(n)}_{I^-\sqcup I^+}\ ,\ L^{(n)})_\alpha\,,$$

where $L^{(n)} = L^{(n)}(S_1, \cdots, S_n|B_1, \cdots, B_{2n})$ is an arbitrary Laurent polynomial which is anti-symmetric in S_i's and symmetric in B_j's.

7. Tower

Final point in the FF axioms is the residue axiom. This leads to a restriction which relates $L^{(n)}$ to $L^{(n-1)}$. We call a set of Laurent polynomials $\{L^{(n)}\}$ which satisfies this restriction for all n a tower. BBS fermions create towers from simpler ones. It is good to generalize $L^{(n)}$ so that it can have non-zero charge. So, here is

Definition 7.1. *We say that* $L^{(\star)} = \{L^{(l,n)}(S_1,\cdots,S_l|B_1,\cdots,B_{2n})\}_{\substack{l,n\geq 0\\ l-n=c}}$ *is a tower of charge* c *if*

$$L^{(l,n)}(S_1,\cdots,S_{l-1},B|B_1,\cdots,B_{2n-2},B,-B)$$
$$= (-1)^c B\prod_{p=1}^{l-1}(B^2-S_p^2)\cdot L^{(l-1,n-1)}(S_1,\cdots,S_{l-1}|B_1,\cdots,B_{2n-2})$$

holds for all $l,n\geq 1$ *with* $l-n=c$.

Before fermions, there are local integrals which create towers by multiplications: $L^{(l,n)} \mapsto f(I,\bar{I})L^{(l,n)}$ where $f(I,\bar{I})$ is an arbitrary polynomial in

$$I_{2j-1,n} = \sum_{k=1}^{2n} B_k^{2j-1}, \quad \bar{I}_{2j-1,n} = \sum_{k=1}^{2n} B_k^{-(2j-1)}\,.$$

Interesting part in the theory of towers come from Q-exact forms, polynomials which vanish identically inside the pairing.

For any Laurent polynomial $Z(S)$ we define

$$D_A[Z](S) = Z(S)P(S) - AZ(SQ)P(-S), \quad Q = q^{\pi i\frac{1-\nu}{\nu}}$$

and call it a Q-exact form. For any Laurent polynomial $\ell(\mathfrak{s})$ we have

$$(\ell, D_A[Z])_\alpha = 0\,.$$

There is an important issue, how to choose the representatives of towers modulo Q-exact forms. There are two ways.

degree restriction: $\quad 0\leq \deg_{S_i} L^{(l,n)}(S_1,\cdots,S_l)\leq 2n-1$

or

parity restriction: $\quad$ only odd powers in S_i's

8. BBS fermions

The tower which corresponds to the primary field Φ_α is given by

$$M_0^{(n)}(S_1,\cdots,S_n)=\langle\Phi_\alpha\rangle\cdot S\wedge S^3\wedge\cdots\wedge S^{2n-1}.$$

This satisfies both degree and parity restrictions.

BBS fermions $\psi_0^*(Z),\chi_0^*(X)$ create towers satisfying the degree restriction.

$$\begin{aligned}&\bigl(\psi_0^*(Z)L^{(\star)}\bigr)^{(l+1,n)}(S_0,\cdots,S_l)\\&\quad=\frac{1}{P(-Z)}\frac{1}{l!}\mathrm{Skew}_{S_0,\cdots,S_l}C(Z,S_0)L^{(l,n)}(S_1,\cdots,S_l)\,,\\&C(S_1,S_2)=\frac{1}{4\nu}S_1\sum_{\epsilon_1,\epsilon_2=\pm}\frac{P(\epsilon_1S_1)P(\epsilon_2S_2)}{\epsilon_1S_1+\epsilon_2S_2},\ P(S)=\prod_{j=1}^{2n}(S-B_j)\\&\bigl(\chi_0^*(Z)L^{(\star)}\bigr)^{(l-1,n)}(S_1,\cdots,S_{l-1})\\&\quad=\frac{1}{P(-Z)}\frac{1}{2}\left(L^{(l,n)}(Z,S_1,\cdots,S_{l-1})-L^{(l,n)}(-Z,S_1,\cdots,S_{l-1})\right).\end{aligned}$$

One can write their action on $M^{(\star)}$ in the determinant form.

$$\begin{aligned}&(\psi^*(Z_1)\cdots\psi^*(Z_k)\chi^*(X_k)\cdots\chi^*(X_1)M^{(\star)})^{(n)}(S_1,\ldots,S_n)\\&=\langle\Phi_\alpha\rangle\frac{(-1)^k}{\prod_{j=1}^{k}\sqrt{P(Z_j)P(-Z_j)}\ \prod_{j=1}^{k}\sqrt{P(X_j)P(-X_j)}}\\&\quad\times\det\begin{pmatrix}0&\cdots&0&C(Z_1,S_1)&\cdots&C(Z_1,S_n)\\\vdots&&\vdots&\vdots&&\vdots\\0&\cdots&0&C(Z_k,S_1)&\cdots&C(Z_k,S_n)\\X_1&\cdots&X_k&S_1&\cdots&S_n\\X_1^3&\cdots&X_k^3&S_1^3&\cdots&S_n^3\\\vdots&&\vdots&\vdots&&\vdots\\X_1^{2n-1}&\cdots&X_k^{2n-1}&S_1^{2n-1}&\cdots&S_n^{2n-1}\end{pmatrix}\end{aligned}$$

9. BBS fermions (continued)

Modified BBS fermions $\psi^*(Z), \bar{\psi}^*(X), \chi^*(Z), \bar{\chi}^*(X)$ create towers with only odd powers in S_j's. They have the Fourier mode expansions in odd degrees.

$$\psi^*(Z) = \sum_{j=1}^{\infty} Z^{-2j+1}\psi^*_{2j-1}, \quad \chi^*(X) = \sum_{j=1}^{\infty} X^{-2j+1}\chi^*_{2j-1}\,,$$

$$\bar{\psi}^*(Z) = \sum_{j=1}^{\infty} Z^{2j-1}\bar{\psi}^*_{2j-1}, \quad \bar{\chi}^*(X) = \sum_{j=1}^{\infty} X^{2j-1}\bar{\chi}^*_{2j-1}\,.$$

We change $\psi_0(Z)$ to $\psi^*(Z)$ by adding Q-exact forms containing degrees more than $2n$, and to $\bar{\psi}^*(Z)$ by adding Q-exact forms containing degrees less than 0. We also do some Bogolubov transformation as a fine tuning.

The primary field tower $S \wedge S^3 \wedge \cdots \wedge S^{2n-1}$ can be considered as Fermi zone. Roughly speaking the operator χ^*_{2j-1} creates a hole at $S^{2n-2j+1}$, $\bar{\chi}^*_{2j-1}$ a hole at S^{2j-1}, ψ^*_{2j-1} a particle at $S^{2n+2j-1}$, and $\bar{\psi}^*_{2j-1}$ a particle at S^{-2j+1}.

Recall that the multiplication of $S_1 \cdots S_{2n}$ amounts to the shift $\alpha \to \alpha + \frac{1-\nu}{\nu}$. Moreover, we have

$$\begin{aligned}
&\psi^*_1 \cdots \psi^*_{2m-1}\bar{\chi}^*_{2m-1} \cdots \bar{\chi}^*_1 \; M^{(\star)}_0 \\
&\quad = \frac{\langle \Phi_\alpha \rangle}{\langle \Phi_{\alpha+2m\frac{1-\nu}{\nu}} \rangle} \cdot (\tfrac{i}{\nu})^m \prod_{j=1}^{m} \cot \tfrac{\pi}{2\nu}(\alpha\nu + (2j-1)) \; M^{(\star)}_m\,,
\end{aligned}$$

$$M^{(n)}_m = \langle \Phi_{\alpha+2m\frac{1-\nu}{\nu}} \rangle \cdot S^{2m+1} \wedge S^{2m+2} \wedge \cdots \wedge S^{2n+2m-1} \prod_{j=1}^{2n} B_j^{-m}.$$

With all these we conclude that the space of descendants of $M^{(\star)}_0$ by fermions, together with the action of the local integrals of motion, has the same character as that of the space of fields in CFT:

$$\bigoplus_{m=-\infty}^{\infty} \mathcal{V}_{\alpha+2\frac{1-\nu}{\nu}m} \otimes \overline{\mathcal{V}}_{\alpha+2\frac{1-\nu}{\nu}m}\,,$$

where $\mathcal{V}_\alpha$ denotes the Virasoro Verma module with highest weight

$$\Delta_\alpha = \frac{\nu^2}{4(1-\nu)}\alpha(\alpha-2).$$

10. Computation of ω

As we discussed already a new feature in the computation of ω with solitons, is the $\binom{2n}{n}$-fold degeneracy of the eigenvalues of the transfer matrix. The computation goes in three steps.

Step 1 Solving the DDV equation in the limit $\mathbf{n} \to \infty$.

The input from the solitons is the positions of holes $\xi_h = \xi_j = B_j^\nu$ $(j = 1, \dots, 2n)$. The equation becomes linear inhomogeneous integral equation.

$$\log \mathfrak{a}(\zeta) - \int_0^\infty K(\zeta/\xi) \log \mathfrak{a}(\xi) \frac{d\xi^2}{\xi^2}$$
$$= \log \left(\frac{d(\zeta)}{a(\zeta)} \right) - 2\pi i \nu \kappa - \sum_h \Phi(\zeta/\xi_h) + \sum_c \Phi(\zeta/\xi_c) \, .$$

The conclusion is the explicit formula for

$$\rho(\zeta) = \frac{P(Z)}{P(-Z)}.$$

Step 2 Solving the Boos-Göhmann equation

The dimension of the solution space of the homogeneous equation is $\binom{2n}{n}$.

$$G(\zeta, \xi) = \delta_\xi^- \psi_0(\zeta/\xi, \alpha) + \frac{1}{2\pi i} \int_{\mathbb{R}_+ e^{+i0}} \bigl(\psi_0(q\zeta/\eta, \alpha) - \psi_0(q^{-1}\zeta/\eta, \alpha)\bigr) \, G(\eta, \xi) \frac{d\eta^2}{\eta^2 \rho(\eta)} \, .$$

For any $\ell^{(n)} \in \mathbb{P}(\wedge^n \mathbb{C}^{2n})$ we have a solution

$$G(\zeta, \xi; \ell^{(n)}) = \frac{(\ell^{(n)}, R_{Z,X}^{(n)})_\alpha}{(\ell^{(n)}, M_0^{(n)})_\alpha},$$

$$R_{Z,X}^{(n)}(S_1, \cdots, S_n) = -\frac{1}{\nu} \frac{ZX}{Z^2 - X^2} \frac{P(Z)}{P(-X)} \prod_{p=1}^n \frac{X^2 - S_p^2}{Z^2 - S_p^2} M_0^{(n)}(S_1, \dots, S_n).$$

Two theories are merging!

11. Computation of ω (continued)

Step 3 Substitution in the formula for ω

$$\omega(\zeta,\xi)=\delta^-_\zeta\delta^-_\xi\Delta^{-1}_\zeta\psi_0(\zeta/\xi,\alpha)+\frac{1}{2\pi i}\int\limits_{\mathbb{R}_+e^{+i0}}\delta^-_\zeta\psi_0(\zeta/\eta,\alpha)G(\eta,\xi)\frac{d\eta^2}{\eta^2\rho(\eta)}\,.$$

Result is remarkable.

$$\omega(\zeta,\xi;\ell^{(n)})=\frac{(\ell^{(n)},L^{(n)}_{Z,X})_\alpha}{(\ell^{(n)},M^{(n)}_0)}\,,$$

where the polynomial $L^{(n)}_{Z,X}$ is given by

$$L^{(n)}_{Z,X}(S_1,\cdots,S_n)=\frac{1}{P(-Z)P(-X)}\begin{vmatrix}0 & C(Z,S_1) & \cdots & C(Z,S_n)\\ X & S_1 & \cdots & S_n\\ X^3 & S_1^3 & \cdots & S_n^3\\ \vdots & \vdots & \vdots & \\ X^{2n-1} & S_1^{2n-1} & \cdots & S_n^{2n-1}\end{vmatrix}.$$

Merci!