Form factor approach to correlation functions of critical models

long-distance asymptotic behavior of two-point correlation functions in the XXZ Heisenberg chain

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The spin-1/2 XXZ Heisenberg chain

The XXZ spin-1/2 Heisenberg chain is a quantum interacting model defined on a one-dimensional lattice with M sites, with Hamiltonian,

$$H_{XXZ} = \sum_{m=1}^{M} \{\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1)\} - h \sum_{m=1}^{M} \sigma_m^z$$

Space of states: $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$ with $\mathcal{H}_m \cong \mathbb{C}^2$

$\sigma_m^{x,y,z} \in \mathrm{End}\mathcal{H}_m$: local spin-1/2 operators (Pauli matrices) at site m

$\Delta \in \mathbb{C}$: anisotropy parameter; $h \in \mathbb{R}$: magnetic field

- periodic boundary conditions
- In the thermodynamic limit $M \to +\infty$ and in the disordered regime, $|\Delta| < 1$ ($\Delta = \cos\zeta,\ 0 < \zeta < \pi$) and $0 < h < h_c$, the spectrum is gapless

Two-point correlation functions as sum over form factors

$$\langle \sigma_1^s \, \sigma_{m+1}^{s'} \rangle = \sum_{|\,\psi'\,\rangle} \mathcal{F}^{(s)}_{\psi_g \psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi' \psi_g}(m+1) \quad \text{with} \quad \mathcal{F}^{(s)}_{\psi\psi'}(m) = \frac{\langle\, \psi \,|\sigma_m^s|\, \psi'\,\rangle}{||\psi||\cdot||\psi'||}$$

Eigenstates parametrized by solutions of logarithmic Bethe equations:

$$\widehat{\xi}(|\{\mu_\ell\}) = \frac{p_0()}{2\pi} - \frac{1}{2\pi M}\sum_{k=1}^{N}\theta(-\mu_{\ell_k}) + \frac{N+1}{2M} = \frac{\ell_j}{M}, \quad j = 1,\ldots,N$$

$p_0(\lambda) = i \log \frac{\sinh(\frac{i\zeta}{2}+\lambda)}{\sinh(\frac{i\zeta}{2}-\lambda)}$ (bare momentum); $\theta(\lambda) = i\log\frac{\sinh(i\zeta+\lambda)}{\sinh(i\zeta-\lambda)}$ (bare phase)

$\ell_j \in \mathbb{Z}$; $\widehat{\xi}(\lambda|\{\mu_\ell\})$: counting function associated to the set of roots $\{\mu_\ell\}$

- **ground state** $|\,\psi_g\,\rangle$**:** $N = N_0$ fixed by h, $\ell_j = j$, $j = 1,\ldots,N_0$
  all roots λ_j are real and densely fill a symmetric interval $[-q,q]$ (the Fermi zone) in the thermodynamic limit $M\to\infty$ with density $\rho(\lambda)$
- **excited states** $|\,\psi'\,\rangle$ ($N = N_0$ for σ^z or N_0+1 for σ^-) are of two types:
  - particle-hole excited states : all roots μ_{ℓ_j} are real
    $\ell_j = j$ for $j\in\{1,\ldots,N\}\setminus\{h_1,\ldots,h_n\}$ and $\ell_{h_j} = p_j \notin \{1,\ldots,N\}$
    $\rightsquigarrow$ particle/hole rapidities: $\widehat{\xi}(\mu_{p_a}|\{\mu_\ell\}) = \frac{p_a}{M}$, $\widehat{\xi}(\mu_{h_a}|\{\mu_\ell\}) = \frac{h_a}{M}$
    $\rightsquigarrow$ 'background' rapidities: $\widehat{\xi}(\mu_j|\{\mu_\ell\}) = \frac{j}{M}$, $j\in\{1,\ldots,N\}$
    $\rightsquigarrow$ shift function: $\mu_j - \lambda_j \underset{M\to\infty}{\sim} \frac{F(\lambda_j)}{M\rho(\lambda_j)}$
  - complex solutions : some of the μ_{ℓ_j} are not real (boundstates) ...

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Particle-hole form factors

- Finite-size form factors can be exactly computed as determinants of elementary functions:

  $$\langle \psi' | \sigma_m^s | \psi_g \rangle = e^{im\mathcal{P}_{\text{ex}}} \cdot \det_N \Omega^{(s)}(\{\lambda\},\{\mu\})$$

  with $|\psi_g\rangle \equiv |\psi(\{\lambda\})\rangle$, $|\psi'\rangle \equiv |\psi(\{\mu\})\rangle$

- dependence in distance m in phase factor given by relative momentum:

  $$\mathcal{P}_{\text{ex}} = \sum_{j=1}^{N} p_0(\mu_{\ell_j}) - \sum_{j=1}^{N_0} p_0(\lambda_j) = \frac{2\pi}{M} \sum_{k=1}^{n} (p_k - h_k)$$

  labelled in terms of particle/hole integers

- singularities at $\lambda_j = \mu_{\ell_k}$ $\rightsquigarrow$ factorize out Cauchy determinant:

  $$\det_N \Omega^{(s)}(\{\lambda\},\{\mu\}) = \det_N \left[\frac{1}{\sinh(\mu_{\ell_k} - \lambda_j)} \right] \cdot \underbrace{\det_N \widetilde{\Omega}^{(s)}(\{\lambda\},\{\mu\})}_{\text{non singular}}$$

Thermodynamic limit of particle-hole form factors

1. Non-singular part $\longrightarrow$ smooth thermodynamic limit $\mathcal{S}(\{\mu_p\};\{\mu_h\})[F]$

$$\sum_{j=1}^{N}[f(\mu_{\ell_j}) - f(\lambda_j)] = \sum_{j=1}^{n}[f(\mu_{p_j}) - f(\mu_{h_j})] + \sum_{j=1}^{N}[f(\mu_j) - f(\lambda_j)]$$
$$\underset{M\to\infty}{\longrightarrow} \sum_{j=1}^{n}[f(\mu_{p_j}) - f(\mu_{h_j})] + \int_{-q}^{q} f'(\lambda)F(\lambda)\,d\lambda$$

2. Singular part (Cauchy)

   - idea: multiply and divide by counting function $\widehat{\xi}$

$$\frac{1}{\sinh(\lambda-\mu)} = \underbrace{\frac{\widehat{\xi}(\lambda)-\widehat{\xi}(\mu)}{\sinh(\lambda-\mu)}}_{\substack{\text{smooth function}\\ cf.1}} \times \underbrace{\frac{1}{\widehat{\xi}(\lambda)-\widehat{\xi}(\mu)}}_{\substack{\propto\text{ difference of integers}\\ \text{(possibly shifted by shift function)}}}$$

   - (some of) the products over 'differences of integers' are written as products over ratios of Gamma functions

ex: $\displaystyle\prod_{k=1}^{n}\frac{\Gamma(h_k+F(\mu_{h_k}))\,\Gamma(N+1-h_k-F(\mu_{h_k}))}{\Gamma(h_k)\,\Gamma(N+1-h_k)}, \quad h_k\in\{1,\dots,N\}$

$\rightsquigarrow$ approximated by Stirling formula only if particle/hole integers macroscopically away from the Fermi boundaries

$\rightsquigarrow$ large-size behavior depends on the number of particles/holes collapsing on the Fermi surface

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$$\xrightarrow[M\to\infty]{}\sum_{j=1}^{n}[f(\mu_{p_j})-f(\mu_{h_j})]+\int_{-q}^{q}f'(\lambda)F(\lambda)\,d\lambda$$

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ex: $\widehat{\xi}(\mu_j)-\widehat{\xi}(\mu_k)=\frac{1}{M}(j-k)$

$\widehat{\xi}(\mu_j)-\widehat{\xi}(\lambda_k)\sim\frac{1}{M}(j-k+F(\lambda_k))$

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- 1srt example: particles and holes away from the Fermi boundaries

$$\lim_{N,M\to\infty} \frac{\Gamma(p_k - N + F(\mu_{p_k}))\,\Gamma(p_k)\,\Gamma(N+1-h_k-F(\mu_{h_k}))\,\Gamma(h_k+F(\mu_{h_k}))}{\Gamma(p_k-N)\,\Gamma(p_k+F(\mu_{p_k}))\,\Gamma(N+1-h_k)\,\Gamma(h_k)}$$

$$= \lim_{N,M\to\infty} \left(\frac{p_k - N}{p_k}\right)^{F(\mu_{p_k})} \left(\frac{N-h_k}{h_k}\right)^{-F(\mu_{h_k})}$$

$$= \lim_{N,M\to\infty} \left(\frac{\widehat{\xi}(\mu_{p_k}) - \widehat{\xi}(q)}{\widehat{\xi}(\mu_{p_k}) - \widehat{\xi}(-q)}\right)^{F(\mu_{p_k})} \left(\frac{\widehat{\xi}(q) - \widehat{\xi}(\mu_{h_k})}{\widehat{\xi}(\mu_{h_k}) - \widehat{\xi}(-q)}\right)^{-F(\mu_{h_k})}$$

$$= \left(\frac{p(\mu_{p_k}) - p(q)}{p(\mu_{p_k}) - p(-q)}\right)^{F(\mu_{p_k})} \left(\frac{p(q) - p(\mu_{h_k})}{p(\mu_{h_k}) - p(-q)}\right)^{-F(\mu_{h_k})}$$

(since $\widehat{\xi}(\lambda) \to \frac{p(\lambda)+p(q)}{2\pi}$ with $p(\lambda) = 2\pi \int_0^\lambda \rho(\mu)\, d\mu$: dressed momemtum)

$\rightsquigarrow$ smooth function of μ_{p_k}, μ_{h_k}

- 2nd example: critical state of $\mathbf{P}_\ell$ class

particles and holes are all on the Fermi boundaries

$n_p^\pm$ particles, resp. $n_h^\pm$ holes, with rapidities equal to $\pm q$ such that

$$n_p^+ + n_p^- = n_h^+ + n_h^- = n, \qquad n_p^+ - n_h^+ = n_h^- - n_p^- = \ell, \qquad \ell \in \mathbb{Z}$$

Define $p_j = p_j^+ + N$ if $\mu_{p_j} = q$, $\quad p_j = 1 - p_j^-$ if $\mu_{p_j} = -q$

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$$= \left(\frac{p(\mu_{p_k}) - p(q)}{p(\mu_{p_k}) - p(-q)}\right)^{F(\mu_{p_k})} \left(\frac{p(q) - p(\mu_{h_k})}{p(\mu_{h_k}) - p(-q)}\right)^{-F(\mu_{h_k})}$$

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Define $p_j = p_j^+ + N$ if $\mu_{p_j} = q$, $p_j = 1 - p_j^-$ if $\mu_{p_j} = -q$

$h_j = N + 1 - h_j^+$ if $\mu_{h_j} = q$, $h_j = h_j^-$ if $\mu_{h_j} = -q$

$$\prod_{k=1}^{n} \frac{\Gamma(p_k - N + F(\mu_{p_k}))\,\Gamma(p_k)\,\Gamma(N+1-h_k - F(\mu_{h_k})\,\Gamma(h_k + F(\mu_{h_k}))}{\Gamma(p_k - N)\,\Gamma(p_k + F(\mu_{p_k}))\,\Gamma(N+1-h_k)\,\Gamma(h_k)}$$
$$\sim N^{-\ell[F(q)+F(-q)]} \prod_{k=1}^{n_p^+} \frac{\Gamma(p_k^+ + F(q))}{\Gamma(p_k^+)} \prod_{k=1}^{n_p^-} \frac{\Gamma(p_k^- + F(-q))}{\Gamma(p_k^-)} \prod_{k=1}^{n_h^+} \frac{\Gamma(h_k^+ - F(q))}{\Gamma(h_k^+)} \prod_{k=1}^{n_h^-} \frac{\Gamma(h_k^- + F(-q))}{\Gamma(h_k^-)}$$

$\rightsquigarrow$ decreasing exponent is modified + discrete structure in finite part

Thermodynamic limit of particle-hole critical form factors

Inside a given $\mathbf{P}_\ell$ class :

$\rightsquigarrow$ phase factors $\mathcal{P}_{\text{ex}}$ depend on the particular state we consider:

$$\mathcal{P}_{\text{ex}} = \frac{2\pi}{M} \sum_{k=1}^{n} (p_k - h_k) = 2\ell k_F + \frac{2\pi}{M} \mathcal{P}_{\text{ex}}^{(d)} \qquad (k_F\text{: Fermi momentum } p(q))$$

$$\mathcal{P}_{\text{ex}}^{(d)} = \sum_{j=1}^{n_p^+} (p_j^+ - 1) + \sum_{j=1}^{n_h^+} h_j^+ - \sum_{j=1}^{n_p^-} p_j^- - \sum_{j=1}^{n_h^-} (h_j^- - 1)$$

$\rightsquigarrow$ smooth parts $\mathcal{S}(\{\mu_p\};\{\mu_h\})[F]$ are all the same

$\rightsquigarrow$ critical exponents θ_ℓ are all the same

$\rightsquigarrow$ finite discrete parts depend on the particular state we consider (they are expressed in terms of particle/hole integers around the Fermi zone)

$\hookrightarrow$ all critical form factors inside a same $\mathbf{P}_\ell$ class can be expressed in terms of the simplest form factor of the class (the ℓ-shifted state $|\,\psi_\ell\,\rangle$ with integers $\ell_j = j + \ell$) by just taking in consideration the modification of the discrete part

$$\mathcal{F}^{(s)}_{\psi_g\,\psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi'\,\psi_g}(m+1) \sim e^{2im\ell k_F}\, e^{\frac{2\pi i m}{M}\mathcal{P}_{\text{ex}}^{(d)}}\, M^{-\theta_\ell^{(ss')}} \left[\mathcal{F}^{(s)}_{\psi_g\,\psi'} \cdot \mathcal{F}^{(s')}_{\psi'\,\psi_g}\right]_{\text{finite}}$$

with $\theta_\ell = (F_+ + \ell)^2 + (F_- + \ell)^2$

and

$$\left[\mathcal{F}^{(s)}_{\psi_g\,\psi'} \cdot \mathcal{F}^{(s')}_{\psi'\,\psi_g}\right]_{\text{finite}} = \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell} \cdot \mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\text{finite}} \cdot \frac{G^2(1+F_+)G^2(1-F_-)}{G^2(1+\ell+F_+)G^2(1-\ell-F_-)}$$

$$\times R_{n_p^+,n_h^+}(\{p^+\},\{h^+\}|F_+)\; R_{n_p^-,n_h^-}(\{p^-\},\{h^-\}|-F_-)$$

Thermodynamic limit of particle-hole critical form factors

$\hookrightarrow$ all critical form factors inside a same $\mathbf{P}_\ell$ class can be expressed in terms of the simplest form factor of the class (the ℓ-shifted state $|\,\psi_\ell\,\rangle$ with integers $\ell_j = j+\ell$) by just taking in consideration the modification of the discrete part

$$\mathcal{F}^{(s)}_{\psi_g\,\psi'}(1)\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}(m+1) \sim e^{2im\ell k_F}\, e^{\frac{2\pi i m}{M}\mathcal{P}^{(d)}_{\rm ex}}\, M^{-\theta^{(ss')}_\ell}\left[\mathcal{F}^{(s)}_{\psi_g\,\psi'}\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}\right]_{\rm finite}$$

with $\theta_\ell = (F_+ + \ell)^2 + (F_- + \ell)^2$

and

$$\left[\mathcal{F}^{(s)}_{\psi_g\,\psi'}\cdot\mathcal{F}^{(s')}_{\psi'\,\psi_g}\right]_{\rm finite} = \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell}\cdot\mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\rm finite}\cdot\frac{G^2(1+F_+)G^2(1-F_-)}{G^2(1+\ell+F_+)G^2(1-\ell-F_-)}$$
$$\times R_{n_p^+,n_h^+}(\{p^+\},\{h^+\}|F_+)\; R_{n_p^-,n_h^-}(\{p^-\},\{h^-\}|-F_-)$$

here $\quad F_+ = F(q) + N - N_0, \quad F_- = F(-q)$

$G(z)$ is the Barnes function satisfying $G(z+1) = \Gamma(z)G(z)$

and

$$R_{n,n'}(\{p\},\{h\}|F) = \frac{\prod\limits_{j>k}^{n}(p_j-p_k)^2\prod\limits_{j>k}^{n'}(h_j-h_k)^2}{\prod\limits_{j=1}^{n}\prod\limits_{k=1}^{n'}(p_j+h_k-1)^2}\left[\frac{\sin(\pi F)}{\pi}\right]^{2n'}$$
$$\times\prod_{k=1}^{n}\frac{\Gamma^2(p_k+F)}{\Gamma^2(p_k)}\prod_{k=1}^{n'}\frac{\Gamma^2(h_k-F)}{\Gamma^2(h_k)}$$

Summation over critical form factors

In the thermodynamic limit:

$$\langle \sigma_1^s \, \sigma_{m+1}^{s'} \rangle_{\text{cr}} = \lim_{M\to\infty} \sum_{\ell=-\infty}^{\infty} \sum_{|\,\psi'\,\rangle \text{ in } \mathbf{P}_\ell \text{ class}} \mathcal{F}^{(s)}_{\psi_g \psi'}(1) \cdot \mathcal{F}^{(s')}_{\psi' \psi_g}(m+1)$$

$$= \lim_{M\to\infty} \sum_{\ell=-\infty}^{\infty} e^{2im\ell k_F} \, M^{-\theta_\ell^{(ss')}} \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell} \, \mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\text{finite}} \prod_{\epsilon=\pm} \frac{G^2(1+\epsilon F_\epsilon)}{G^2(1+\epsilon\ell+\epsilon F_\epsilon)}$$

$$\times \underbrace{\sum_{\substack{\{p\},\{h\} \\ n_p^+ - n_h^+ = \ell}} e^{\frac{2\pi i m}{M} \mathcal{P}_{ex}^{(d)}} \prod_{\epsilon=\pm} R_{n_p^\epsilon, n_h^\epsilon}(\{p^\epsilon\},\{h^\epsilon\}|\epsilon F_\epsilon)}_{\substack{\text{sum over all possible configurations of integers} \\ \text{in the } \mathbf{P}_\ell \text{ class}}}$$

$$\sum_{\substack{n_p,n_h=0 \\ n_p-n_h=\ell}}^{\infty} \sum_{\substack{p_1<\cdots<p_{n_p} \\ p_a\in\mathbb{N}^*}} \sum_{\substack{h_1<\cdots<h_{n_h} \\ h_a\in\mathbb{N}^*}} e^{\frac{2\pi i m}{M}\left[\sum_{j=1}^{n_p}(p_j-1)+\sum_{k=1}^{n_h} h_k\right]} R_{n_p,n_h}(\{p\},\{h\}|F)$$

$$= \frac{G^2(1+\ell+F)}{G^2(1+F)} \, \frac{e^{\frac{i\pi m}{M}\ell(\ell-1)}}{\left(1-e^{\frac{2i\pi m}{M}}\right)^{(F+\ell)^2}}$$

Proof

$$\rightsquigarrow \quad \langle \sigma_1^s \, \sigma_{m+1}^{s'} \rangle_{\text{cr}} = \sum_{\ell=-\infty}^{\infty} \frac{e^{2im\ell k_F + i\frac{\pi}{2}[(F_-+\ell)^2-(F_++\ell)^2]}}{(2\pi m)^{\theta_\ell}} \left[\mathcal{F}^{(s)}_{\psi_g\,\psi_\ell} \, \mathcal{F}^{(s')}_{\psi_\ell\,\psi_g}\right]_{\text{finite}}$$

Summation formula: sketch of the proof

Summation formula

$$f_\ell(\nu, w) = w^{\ell(\ell-1)/2} \frac{G^2(1+\ell+\nu)}{G^2(1+\nu)} (1-w)^{-(\nu+\ell)^2}$$

where G is the Barnes G-function: $G(z+1) = \Gamma(z)G(z)$

and $$f_\ell(\nu, w) \equiv \sum_{\substack{n_p, n_h=0 \\ n_p - n_h = \ell}}^{\infty} \sum_{\substack{p_1 < \cdots < p_{n_p} \\ p_a \in \mathbb{N}^*}} \sum_{\substack{h_1 < \cdots < h_{n_h} \\ h_a \in \mathbb{N}^*}} w^{\sum_{j=1}^{n_p}(p_j - 1) + \sum_{k=1}^{n_h} h_k} \left(\frac{\sin \pi\nu}{\pi}\right)^{2n_h}$$
$$\times \frac{\prod_{j>k}^{n_p}(p_j - p_k)^2 \prod_{j>k}^{n_h}(h_j - h_k)^2}{\prod_{j=1}^{n_p} \prod_{k=1}^{n_h}(p_j + h_k - 1)^2} \prod_{j=1}^{n_p} \frac{\Gamma^2(p_j+\nu)}{\Gamma^2(p_j)} \prod_{k=1}^{n_h} \frac{\Gamma^2(h_k - \nu)}{\Gamma^2(h_k)}$$

1. the **case** $\ell \neq 0$ can be obtained from $\ell = 0$ by a "background shift" :

   recast the sum over all possible excitations over the Dirac sea $\mathbb{Z}^-$

   (particles $p_j \in \mathbb{Z}^{+*}$, holes $1 - h_j \in \mathbb{Z}^-$, with $n_p - n_h = \ell$)

   as a sum over excitations over a shifted Dirac sea $\mathbb{Z} \cap]-\infty, \ell]$:

   particles $\ell + \tilde{p}_j$, holes $1 + \ell - \tilde{h}_j$, with $\tilde{n}_p - \tilde{n}_h = 0$

   $$\Longrightarrow \quad f_\ell(\nu, w) = w^{\ell(\ell-1)/2} \frac{G^2(1+\ell+\nu)}{G^2(1+\nu)} f_0(\nu+\ell, w)$$

2. Proof of the **case** $\ell = 0$ details

Summation formula: sketch of the proof (case $\ell = 0$)

Summation formula in the case $\ell = 0$

$$f_0(\nu, w) = (1 - w)^{-\nu^2}$$

where G is the Barnes G-function: $G(z+1) = \Gamma(z) G(z)$

and $$f_0(\nu, w) \equiv \sum_{n=0}^{\infty} \sum_{\substack{p_1 < \cdots < p_n \\ p_a \in \mathbb{N}^*}} \sum_{\substack{h_1 < \cdots < h_n \\ h_a \in \mathbb{N}^*}} w^{\sum_{j=1}^{n}(p_j + h_j - 1)} \left(\frac{\sin \pi\nu}{\pi}\right)^{2n} \times \left[\det_n \frac{1}{p_j + h_k - 1}\right]^2 \prod_{j=1}^{n} \frac{\Gamma^2(p_j + \nu)}{\Gamma^2(p_j)} \frac{\Gamma^2(h_j - \nu)}{\Gamma^2(h_j)}$$

- $f_0(-\nu, w) = f_0(\nu, w)$ $\quad (p_j \leftrightarrow h_j) \quad \rightarrow \quad$ restrict to the case $\nu \geq 0$
- Using the symmetry properties of the sum, one can recast $f_0(\nu, w)$ as the determinant of an infinite matrix:

$$f_0(\nu, w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{h_1, \ldots, h_n = 1}^{\infty} \det_{\substack{j=1,\ldots,n \\ k=1,\ldots,n}} V(h_j, h_k)$$
$$= \det_{\substack{j=1,\ldots,\infty \\ k=1,\ldots,\infty}} [\delta_{jk} + V(j,k)]$$

with

$$V(j,k) = \left(\frac{\sin \pi\nu}{\pi}\right)^2 \frac{\Gamma(j-\nu)\Gamma(k-\nu)}{\Gamma(j)\Gamma(k)} \sum_{p=0}^{\infty} \frac{w^{p+(j+k)/2}}{(p+j)(p+k)} \frac{\Gamma^2(p+1+\nu)}{\Gamma^2(p+1)}$$

Summation formula: sketch of the proof (case $\ell = 0$)

Summation formula in the case $\ell = 0$

$$f_0(\nu, w) = (1-w)^{-\nu^2}$$

where G is the Barnes G-function: $G(z+1) = \Gamma(z)G(z)$

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$$f_0(\nu, w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{h_1,\dots,h_n=1}^{\infty} \det_{\substack{j=1,\dots,n \\ k=1,\dots,n}} V(h_j, h_k) = \det_{\substack{j=1,\dots,\infty \\ k=1,\dots,\infty}} [\delta_{jk} + V(j,k)]$$

with

$$V(j,k) = \left(\frac{\sin \pi\nu}{\pi}\right)^2 \frac{\Gamma(j-\nu)\Gamma(k-\nu)}{\Gamma(j)\Gamma(k)} \sum_{p=0}^{\infty} \frac{w^{p+(j+k)/2}}{(p+j)(p+k)} \frac{\Gamma^2(p+1+\nu)}{\Gamma^2(p+1)}$$

Summation formula: sketch of the proof (case $\ell = 0$, $\nu \in \mathbb{N}$)

- If $\nu = N$ is a positive integer, the determinant becomes finite:
$$f_0(N, w) = \det_{\substack{j=1,\dots,N \\ k=1,\dots,N}} [\delta_{jk} + V(j,k)]$$
with
$$V(j,k) = \frac{w^{(j+k)/2}}{\prod_{\substack{m=1 \\ m\neq j}}^{N}(j-m)\prod_{\substack{m=1 \\ m\neq k}}^{N}(k-m)} \sum_{p=0}^{\infty} \frac{w^p}{(p+j)(p+k)} \prod_{m=1}^{N}(p+m)^2$$
- It can be simplified using the identity:
$$\det(I+V) = \frac{\det(AA^T + AVA^T)}{[\det A]^2} \quad \text{with} \quad A_{jk} = w^{-k/2}\, k^{j-1}$$
and setting $w = e^{-t}$:
$$f_0(N, e^{-t}) = \frac{e^{-tN(N+1)/2}}{\prod_{k=1}^{N-1}(k!)^2} \det_N \left[\partial_t^{j+k-2} \frac{e^{Nt}}{1-e^{-t}} \right]$$
- It can be rewritten as the homogeneous limit of a Cauchy determinant:
$$f_0(N, e^{-t}) = \lim_{\substack{u_1,\dots u_n \to 0 \\ v_1,\dots v_N \to 0}} \frac{e^{-tN(N+1)/2}}{\prod_{j<k}(u_j-u_k)(v_j-v_k)} \det_N \left[\frac{e^{N(t+u_j+v_k)}}{1-e^{-t-u_j-v_k}} \right]$$
Computing the determinant and taking the homogeneous limit, one gets the result. . . QED

Summation formula: sketch of the proof (case $\ell = 0$, $\nu \notin \mathbb{N}$)

- The infinite determinant can be rewritten in terms of Toeplitz and Hankel matrices:

$$f_0(\nu, w) = \det\left[I + T^{-1}[a]\, H[a]\, H[\tilde{b}]\, T^{-1}[b]\right]$$

with $a(z) = \sum_{-\infty}^{\infty} \frac{w^{n/2}}{n+\nu}\, z^n, \qquad b(z) = \sum_{-\infty}^{\infty} \frac{w^{-n/2}}{n-\nu}\, z^n$

and $H_{jk}[a] \equiv [a]_{j+k-1} = -H_{jk}[\tilde{b}] \equiv -[b]_{-j-k+1} = \frac{w^{(j+k-1)/2}}{j+k-1+\nu},$

$$T_{jk}[a] \equiv [a]_{j-k} = \frac{w^{(j-k)/2}}{j-k+\nu}, \qquad T_{jk}[b] \equiv [b]_{j-k} = \frac{w^{-(j-k)/2}}{j-k-\nu}$$

- We use properties of Toeplitz matrices:
  - ⋆ $T[ab] = T[a]\, T[b] + H[a]\, H[\tilde{b}]$

    $\Rightarrow\ f_0(\nu, w) = \det\left[T^{-1}[a]\, T[ab]\, T^{-1}[b]\right]$
  - ⋆ Wiener-Hopf factorization:

$$a(z) = a_+(z)\, a_-(z) \quad \text{with} \quad \begin{cases} a_+(z) = \exp\left\{\sum_{k=1}^{\infty} z^k [\log a]_k\right\} \\ a_-(z) = \exp\left\{\sum_{k=1}^{\infty} z^{-k} [\log a]_{-k}\right\} \end{cases}$$

$$\Rightarrow T[ab] = T[a_-]\, T[a_+ b] = T[a_- b]\, T[a_+]$$

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$$\Rightarrow \quad f_0(\nu, w) = \det\left[T^{-1}[a_+]\, T[a_+ b_-]\, T^{-1}[b_-]\right]$$

$$\text{with} \quad \begin{cases} [\log a]_n = \delta_{n,0} \log \frac{\pi}{\sin \pi\nu} + (1 - \delta_{n,0}) \frac{\nu}{n} w^{n/2} \\ [\log b]_n = \delta_{n,0} \log\left(-\frac{\pi}{\sin \pi\nu}\right) - (1 - \delta_{n,0}) \frac{\nu}{n} w^{n/2} \end{cases}$$

  - $\star$ $\det\left[T^{-1}[a_+]\, T[a_+ b_-]\, T^{-1}[b_-]\right] = \exp\left[\sum_{k=1}^{\infty} k\, [\log a]_k [\log b]_{-k}\right]$ (Widom)

$$\Rightarrow \quad f_0(\nu, w) = \exp\left\{-\sum_{n=1}^{\infty} \frac{\nu^2}{n} w^n\right\} = (1 - w)^{-\nu^2}$$ QED

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Correlation function $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = -\frac{1}{2\pi^2} \, \partial_\alpha^2 \mathbf{D}_m^2 \langle e^{2\pi i \alpha \mathcal{Q}_m} \rangle \big|_{\alpha=0} - 2D + 1$$

where $\mathbf{D}_m^2$ is the second lattice derivative, D is the average density, and

$$\mathcal{Q}_m = \frac{1}{2} \sum_{k=1}^{m} (1 - \sigma_k^z)$$

$\rightsquigarrow$ study form factors $\langle \, \psi_\alpha(\{\mu\}) \, | e^{2\pi i \alpha \mathcal{Q}_m} | \, \psi_g \, \rangle$ where $| \, \psi_\alpha(\{\mu\}) \, \rangle$ is an α-deformed Bethe state, with $\{\mu\}$ solution of

$$M p_0(\mu_{\ell_j}) - \sum_{k=1}^{N} \theta(\mu_{\ell_j} - \mu_{\ell_k}) = 2\pi \Big(\ell_j + \alpha - \frac{N+1}{2} \Big)$$

For the $\mathbf{P}_\ell$ class:

- excitation momentum $2\alpha k_F + \mathcal{P}_{\text{ex}}$
- shift functions $F_\pm$: $F_- = F_+ = \alpha \mathcal{Z} + \ell(\mathcal{Z} - 1)$ with $\mathcal{Z} = Z(\pm q)$
  where $Z(\lambda)$ is the dressed charge given by
  $$Z(\lambda) + \frac{1}{2\pi} \int_{-q}^{q} d\mu \, \frac{\sin 2\zeta}{\sinh(\lambda - \mu + i\zeta) \sinh(\lambda - \mu - i\zeta)} \, Z(\mu) = 1$$
- exponent $\theta_{\alpha+\ell}$: $\theta_{\alpha+\ell} = 2[(\alpha + \ell)\mathcal{Z}]^2$,

Correlation function $\langle \sigma_1^z \, \sigma_{m+1}^z \rangle$

$\rightsquigarrow$ leading asymptotic terms for all oscillating harmonics:

$$\langle e^{2\pi i \alpha \mathcal{Q}_m} \rangle_{cr} = \sum_{\ell=-\infty}^{\infty} |\mathcal{F}_{\alpha+\ell}|^2_{\text{finite}} \, \frac{e^{2im(\alpha+\ell)k_F}}{(2\pi m)^{\theta_{\alpha+\ell}}}$$

with $\quad \theta_{\alpha+\ell} = 2[(\alpha+\ell)\mathcal{Z}]^2,$

and $\quad |\mathcal{F}_{\alpha+\ell}|^2_{\text{finite}} = \lim_{M\to\infty} M^{\theta_{\alpha+\ell}} \, \frac{|\langle \psi_g | \psi_{\alpha+\ell} \rangle|^2}{||\psi_g||^2 \, ||\psi_{\alpha+\ell}||^2},$

where $|\, \psi_{\alpha+\ell} \,\rangle$ is the $(\alpha+\ell)$-shifted ground state

Rm: terms $\ell = 0, \pm 1$ coincide with results from master equation analysis

$\rightsquigarrow$ leading asymptotic terms for the two-point function:

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{\text{cr}} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2 \sum_{\ell=1}^{\infty} |\mathcal{F}_\ell^z|^2_{\text{finite}} \, \frac{\cos(2m\ell k_F)}{(2\pi m)^{2\ell^2 \mathcal{Z}^2}}$$

with $\quad |\mathcal{F}_\ell^z|^2_{\text{finite}} = \lim_{M\to\infty} M^{2\ell^2 \mathcal{Z}^2} \, \frac{|\langle\, \psi_g \,| \sigma_1^z |\, \psi_\ell \,\rangle|^2}{||\psi_g||^2 \, ||\psi_\ell||^2},$

where $|\, \psi_\ell \,\rangle$ is the ℓ-shifted ground state

Correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$

$\leadsto$ critical excited states of the $\mathbf{P}_\ell$ class in the (N_0+1)-sector

- critical values of the shift function in the $\mathbf{P}_\ell$ class:
$$F_- = \ell(\mathcal{Z}-1) - \frac{1}{2\mathcal{Z}}, \qquad F_+ = \ell(\mathcal{Z}-1) + \frac{1}{2\mathcal{Z}}$$
- critical exponents: $\theta_\ell = 2\ell^2\mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}$
- simplest form factor in the $\mathbf{P}_\ell$ class:
$$\left|\mathcal{F}_\ell^+\right|^2_{\text{finite}} = \lim_{M\to\infty} M^{(2\ell^2\mathcal{Z}^2+\frac{1}{2\mathcal{Z}^2})} \frac{|\langle\, \psi_g \,|\sigma_1^+|\, \psi_\ell \,\rangle|^2}{||\psi_g||^2\,||\psi_\ell||^2}$$
where $|\,\psi_\ell\,\rangle$ is the ℓ-shifted ground state in the (N_0+1)-sector

$\leadsto$ leading asymptotic terms for the two-point function:

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle_{cr} = \frac{(-1)^m}{(2\pi m)^{\frac{1}{2\mathcal{Z}^2}}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell \, |\mathcal{F}_\ell^+|^2_{\text{finite}} \, \frac{e^{2im\ell k_F}}{(2\pi m)^{2\ell^2\mathcal{Z}^2}}$$

Results for the XXZ chain

2-point functions

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle_{cr} = (2D-1)^2 - \frac{2\mathcal{Z}^2}{\pi^2 m^2} + 2\sum_{\ell=1}^{\infty} |\mathcal{F}_\ell^z|^2_{\text{finite}} \frac{\cos(2m\ell k_F)}{(2\pi m)^{2\ell^2\mathcal{Z}^2}}$$

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle_{cr} = \frac{(-1)^m}{(2\pi m)^{\frac{1}{2\mathcal{Z}^2}}} \sum_{\ell=-\infty}^{\infty} (-1)^\ell |\mathcal{F}_\ell^+|^2_{\text{finite}} \frac{e^{2im\ell k_F}}{(2\pi m)^{2\ell^2\mathcal{Z}^2}}$$

- $\mathcal{Z} = Z(q)$ where $Z(\lambda)$ is the dressed charge
  $$Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda-\mu)\, Z(\mu) = 1$$
- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu)\mathrm{d}\mu = \frac{1-\langle\sigma^z\rangle}{2} = \frac{k_F}{\pi}$
- $$|\mathcal{F}_\ell^z|^2_{\text{finite}} = \lim_{M\to\infty} M^{2\ell^2\mathcal{Z}^2} \frac{|\langle \psi_g | \sigma_1^z | \psi_\ell \rangle|^2}{\langle \psi_g | \psi_g \rangle \langle \psi_\ell | \psi_\ell \rangle}$$
- $$|\mathcal{F}_\ell^+|^2_{\text{finite}} = \lim_{M\to\infty} M^{\left(2\ell^2\mathcal{Z}^2 + \frac{1}{2\mathcal{Z}^2}\right)} \frac{|\langle \psi_g | \sigma_1^+ | \psi_\ell \rangle|^2}{\langle \psi_g | \psi_g \rangle \langle \psi_\ell | \psi_\ell \rangle}$$

Further results and open questions

- **Further results**
  - Time dependent case for the Bose gas (simpler model: no bound-states)
    ⇝ see Karol's seminar (contribution of a saddle point away from the Fermi surface)
  - Asymptotics for large distances in the temperature case (contact with QTM method)
    ⇝ see Nikita's seminar

- **Some open problems...**
  - Limit $h = 0$:
    - amplitude ?
    - XXX limit ?
  - Time dependent case for XXZ : needs careful treatment of bound-states
  - Other models like Sine-Gordon?